# Green Function Formalism
# and
# Electroweak Nuclear Response

## Lecture 2

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# Outline

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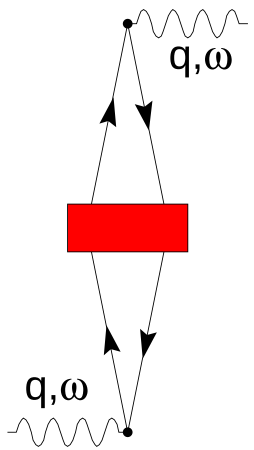
- ⋆ Writing the response in terms of spectral functions
  - ▷ Impulse approximation
  - ▷ Final State Interactions

⋆ Recall

$$
\begin{aligned}
S(\mathbf{q},\omega) &= \sum_n \Big| \sum_k \langle n | a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} | 0 \rangle \Big|^2 \, \delta(\omega + E_0 - E_n) \\
&= \int \frac{dt}{2\pi} \, \mathrm{e}^{i(\omega+E_0)t} \sum_{p,k} \langle 0 | a_{\mathbf{p}+\mathbf{q}} a^\dagger_{\mathbf{p}} \, \mathrm{e}^{-iHt} \, a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} | 0 \rangle
\end{aligned}
$$

$=$ 


⋆ $S(\mathbf{q},\omega)$ can be expressed in terms of interactions and propagators (Green functions) of nucleons in particle and hole states

⋆ Definition of Green function (assume translation invariance)

$$iG(x - x') = \langle 0|T[\psi(x)\psi^\dagger(x')]|0\rangle$$

$$\psi(x) = \sqrt{\frac{1}{V}} \sum_{\mathbf{k}} a_{\mathbf{k}} \mathrm{e}^{-ikx} \ , \ \ \psi^\dagger(x) = \sqrt{\frac{1}{V}} \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}} \mathrm{e}^{ikx}$$

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⋆ After Fourier transformation $(\eta = 0^+)$

$$\begin{aligned} G(\mathbf{k}, E) &= \sum_n \left\{ \frac{|\langle n_{(N+1)}(\mathbf{k}) | a^\dagger_{\mathbf{k}} | 0_N \rangle|^2}{E - (E_n - E_0) + i\eta} + \frac{|\langle n_{(N-1)}(-\mathbf{k}) | a_{\mathbf{k}} | 0_N \rangle|^2}{E - (E_0 - E_n) - i\eta} \right\} \\ &= G_p(\mathbf{k}, E) + G_h(\mathbf{k}, E) \\ &= \int dE' \left[ \frac{P_p(\mathbf{k}, E')}{E - E' + i\eta} + \frac{P_h(\mathbf{k}, E')}{E - E' - i\eta} \right] \end{aligned}$$

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⋆ $P_h$ and $P_p$ are the *spectral functions* of hole and particle states

## $\star$ Definition of spectral functions

$$P_h(\mathbf{k}, E) = \sum_n |\langle n_{(N-1)}(\mathbf{k})|a_{\mathbf{k}}|0_N\rangle|^2 \delta(E - E_0 + E_n) = \frac{1}{\pi}\mathrm{Im}\ G_h(\mathbf{k}, E)$$

$$P_p(\mathbf{k}, E) = \sum_n |\langle n_{(N+1)}(\mathbf{k})|a^\dagger_{\mathbf{k}}|0_N\rangle|^2 \delta(E - E_n + E_0) = -\frac{1}{\pi}\mathrm{Im}\, G_p(\mathbf{k}, E)$$

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- Non interacting system at uniform density $\rho = s k_F^3/6\pi^2$ (Fermi gas)

$$\sum_n \langle n_{(N+1)}|a^\dagger_{\mathbf{k}}|0_N\rangle \longrightarrow \theta(|\mathbf{k}|-k_F)\ ,\ \ E_0-E_n=\epsilon_k=|\mathbf{k}^2|/2m$$

$$\sum_n \langle n_{(N-1)}|a_{\mathbf{k}}|0_N\rangle \longrightarrow \theta(k_F-|\mathbf{k}|)\ ,\ \ E_n-E_0=-\epsilon_k$$

$$P_h(\mathbf{k},E) = \theta(k_F-|\mathbf{k}|)\delta(E+\epsilon^0_k)$$

$$P_p(\mathbf{k},E) = \theta(|\mathbf{k}|-k_F)\delta(E-\epsilon^0_k)$$

⋆ Interacting systems: replace $\epsilon_k^0 \longrightarrow \epsilon_k^0 + \Sigma(\mathbf{k}, E)$

$$G_h(\mathbf{k}, E) = \frac{1}{E + \epsilon_k^0 + \Sigma(\mathbf{k}, E)}$$

⋆ Quasiparticle picture: isolate contributions of $1h$ (i.e. *bound*) intermediate states, having strength

$$Z_k = |\langle -\mathbf{k}|a_{\mathbf{k}}|0\rangle|^2$$

and exhibiting poles at the quasiparticle energies $\epsilon_k = \epsilon_k^0 + \mathrm{Re}\,\Sigma(\mathbf{k}, \epsilon_k)$

⋆ Rewrite the Green function (e.g. for hole states) as

$$G_h(\mathbf{k}, E) = \frac{Z_k}{E + \epsilon_k + iZ_k\,\mathrm{Im}\,\Sigma(\mathbf{k}, e_k)} + G_h^B(\mathbf{k}, E)$$

$G_h^B(\mathbf{k}, E)$ is a smooth contribution, corresponding to $2h-1p$, $3h-2p, \ldots$ intermediate states

⋆ A nonvanishing $G_h^B(\mathbf{k}, E)$ is a signal of correlation effects

## Calculation of the spectral functions

- ⋆ Correlated states obtained from mean field (shell model or Fermi gas) states through the transformation

$$|n\rangle = F|n_{MF}\rangle \quad , \quad F = \mathcal{S} \prod_{j>i} f_{ij}$$

- ⋆ The two-nucleon correlation operator reflects the complexity of the nucleon-nucleon (NN) force (spin-isospin (ST) dependent, non central)

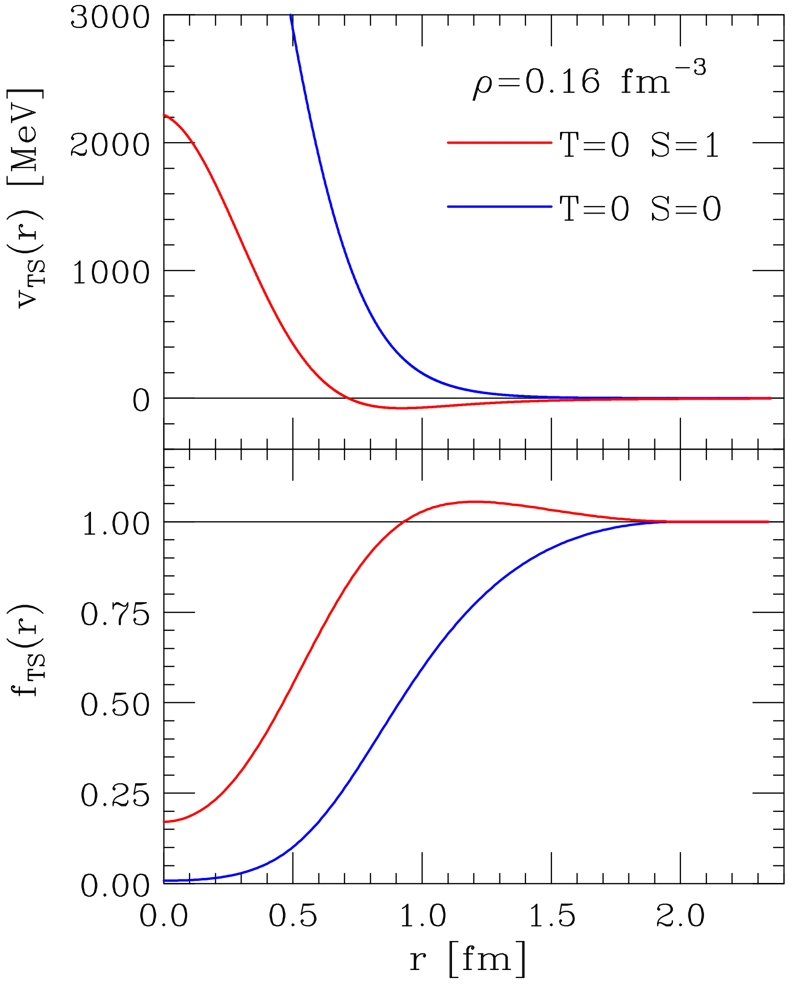
$$f_{ij} = \sum_{TS} \left[ f_{TS}(r_{ij}) + \delta_{S1} f_{Tt}(r_{ij}) S_{ij} \right] P_{TS}$$

$$P_{TS} \ : \ \ \mathrm{spin-isospin\ projectors} \ \ , \ \ S_{ij} = \sigma_i^\alpha \sigma_j^\beta \left( 3 r_{ij}^\alpha r_{ij}^\beta - \delta^{\alpha\beta} \right)$$

- ⋆ Shapes of $f_{TS}$, $f_{tT}$ determined from minimization of ground state energy

# Shape of central correlation functions

- ANL $v_8'$ potential



# Correlated Basis Function (CBF) perturbation theory

- Split the hamiltonian according to

$$H = H_0 + H_1$$

$$\langle m|H_0|n\rangle = \delta_{mn}\langle m|H|n\rangle \quad , \quad \langle m|H_1|n\rangle = (1-\delta_{mn})\langle m|H|n\rangle$$

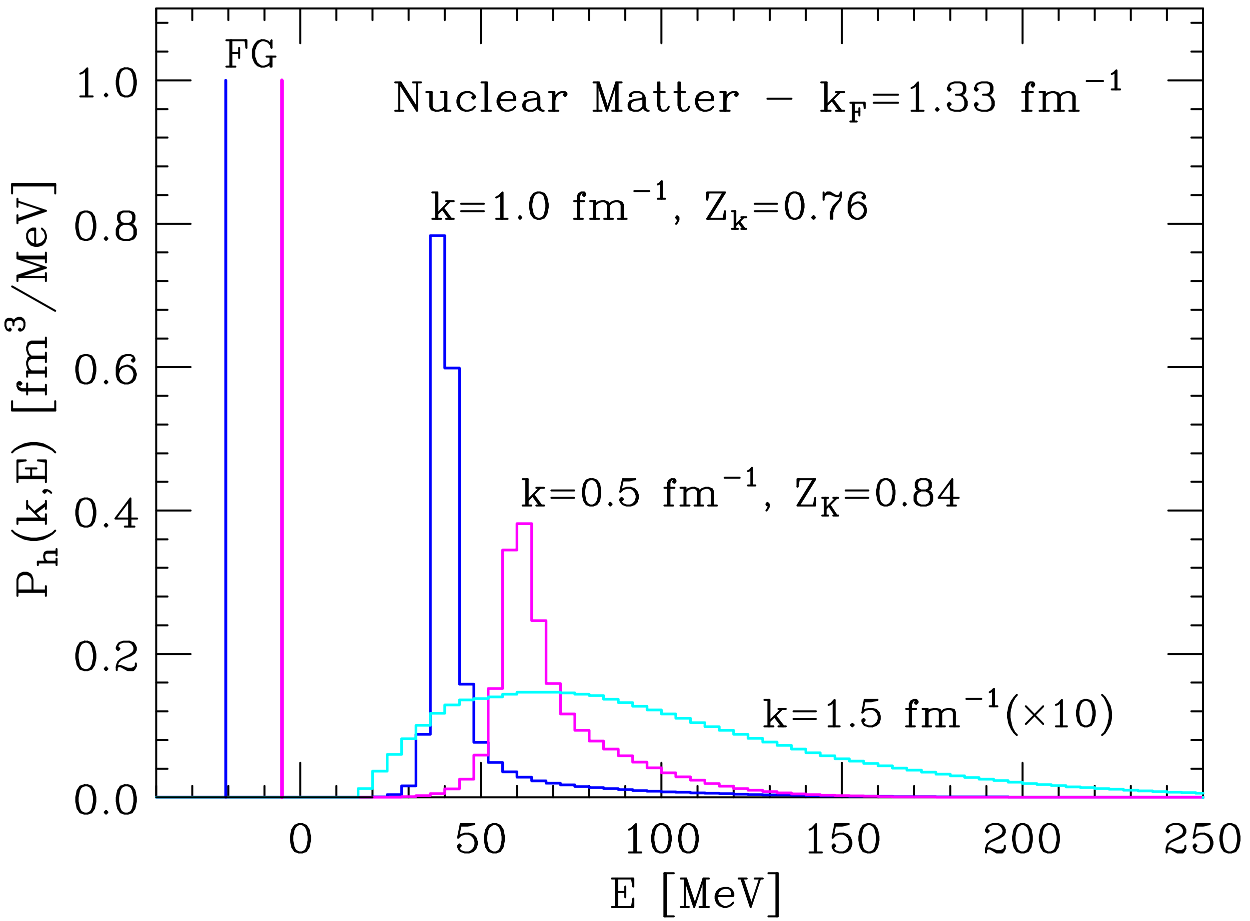
- If correlated states have large overlaps with the eigenstates of the hamiltonian, the matrix elements of $H_1$ are small and perturbation theory can be used to obtain, e.g., the ground state from

$$|\widetilde{0}\rangle = \sum_m (-)^m \left( \frac{H_1 - \Delta E_0}{H_0 - E_0^V} \right)^m |0\rangle$$

$$\Delta E_0 = E_0^V - E_0 = \langle 0|H|0\rangle - E_0$$

## ⋆ Hole spectral function of nuclear matter obtained from CBF

$$P_h(\mathbf{k}, E) = \frac{1}{\pi} \frac{Z_k^2 \, \Sigma(\mathbf{k}, \epsilon_k)}{[E - \epsilon_k^0 - \Sigma(\mathbf{k}, E)]^2 + [Z_k \mathrm{Im}\, \Sigma(\mathbf{k}, \epsilon_k)]^2} + P_h^B(\mathbf{k}, E)$$



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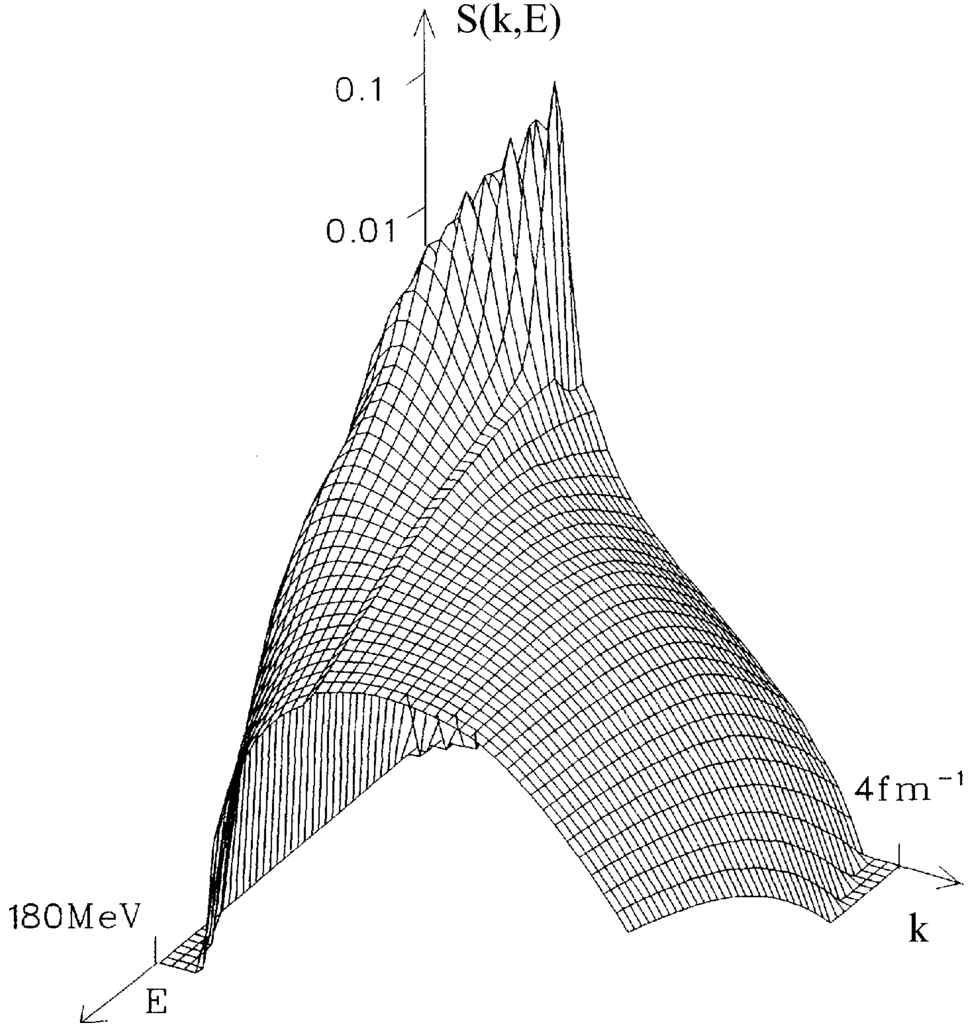
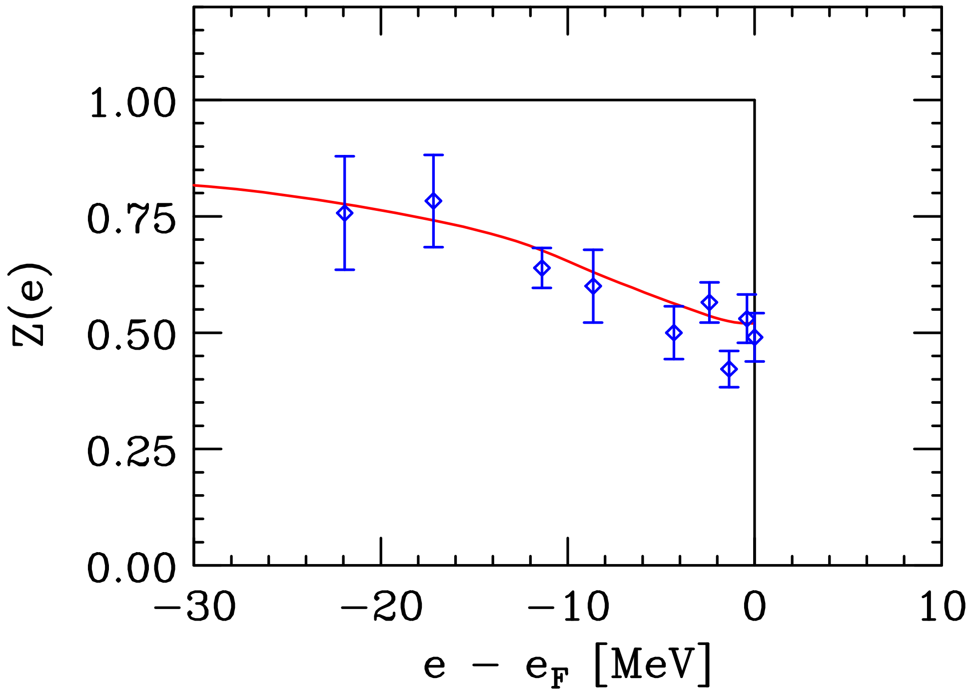


## Experimental evidence of correlation effects

- ⋆ Energy dependence of the spectroscopic strengths of shell model states of $^{208}Pb$, measured in high resolution $(e, e'p)$




- ⋆ Recall: the shell model predicts $Z(e) \equiv 1$

## The impulse approximation (IA)

- Consider the first contribution to the DRPA series. The corresponding response reads

$$S(\mathbf{q},\omega) = \int d^3k \, dE \, P_h(\mathbf{k},E) P_p(\mathbf{k}+\mathbf{q},\omega - E)$$

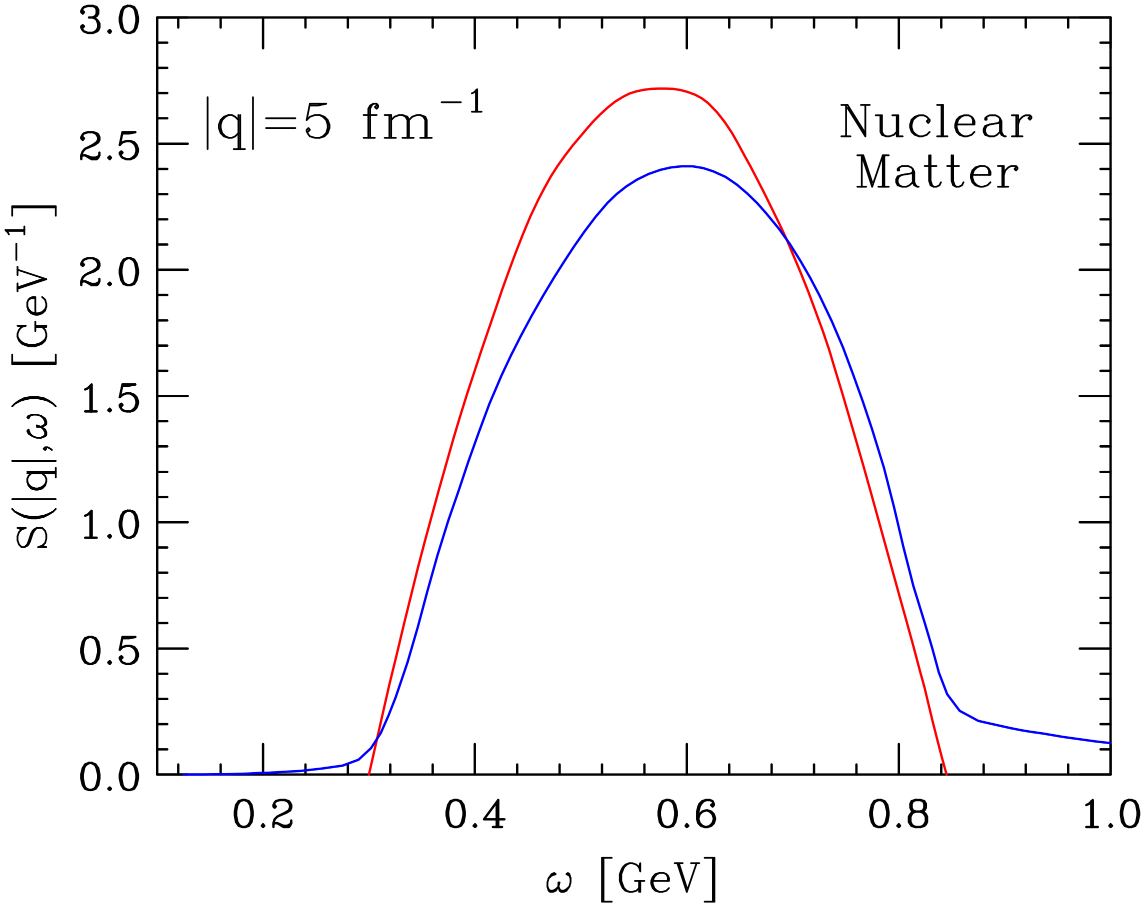
- the particle and hole spectral functions describe initial and final state effects, respectively
- Neglecting all interactions (Fermi gas model)

$$S(\mathbf{q},\omega) = \int d^3k \, \theta(k_F - |\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}| - k_F)\delta(\omega + e^0_k - e^0_{|\mathbf{k}+\mathbf{q}|})$$

- Neglecting final state interactions (FSI)

$$S(\mathbf{q},\omega) = \int d^3k \, dE \, P_h(\mathbf{k},E)\theta(|\mathbf{k}+\mathbf{q}| - k_F)\delta(\omega - e^0_{|\mathbf{k}+\mathbf{q}|} - E)$$

⋆ Effects of *initial state* correlations on the response of nuclear matter at equilibrium density




⋆ NOTE: the FG hole spectral function is shifted in such a way as to reproduce the minumum nuclear matter binding energy

# Including FSI

⋆ Bottom line: FSI do not affect the total inclusive strength. Hence rewrite

$$S(\mathbf{q},\omega) = \int d\omega' S_0(\mathbf{q},\omega') f_{|\mathbf{q}|}(\omega - \omega')$$

where

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- ⋆ At large momentum transfer $|\mathbf{k}+\mathbf{q}| \sim |\mathbf{q}|$ and the folding function is simply related to the particle spectral function through

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- ⋆ Problem: $P_p(\mathbf{k}+\mathbf{q},\omega - E)$ cannot be obtained from non relativistic NMBT

## Approximations for the particle spectral function

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1. energy shift due to the mean field of the spectators
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- ⋆ High energy approximation

1. the struck nucleon moves along a straight trajectory with constant velocity
2. the fast struck nucleon “sees” the spectator system as a collection of fixed scattering centers.

⋆ The *folding function* $f_{|\mathbf{q}|}(\omega)$ is obtained Fourier transforming the eikonal propagator of the struck particle, travelling in the direction of the $z$-axis with velocity $v$

$$f_{|\mathbf{q}|}(\omega) = \int \frac{dt}{2\pi}\, \mathrm{e}^{i\omega t}\, \mathrm{e}^{i \int_0^t dt' \widetilde{V}_{|\mathbf{q}|}(vt')} \quad , \quad \widetilde{V}_{|\mathbf{q}|}(z) = \langle\, \frac{1}{A} \sum_{j>i} \Gamma_{|\mathbf{q}|}(\mathbf{r}_{ij} + z)\, \rangle$$

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- ⋆ $\Gamma_{|\mathbf{q}|}$ is the Fourier transform of the NN scattering amplitude

$$A_{|\mathbf{q}|}(k) = \frac{|\mathbf{q}|}{4\pi}\sigma(i+\alpha)\mathrm{e}^{-\beta k^2}$$

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- ⋆ neglecting all NN correlations, $\widetilde{V}_{|\mathbf{q}|}(z) \to \widetilde{V}^0_{|\mathbf{q}|} = v\rho\sigma(i+\alpha)/2$, and the quasiparticle approximation is recovered

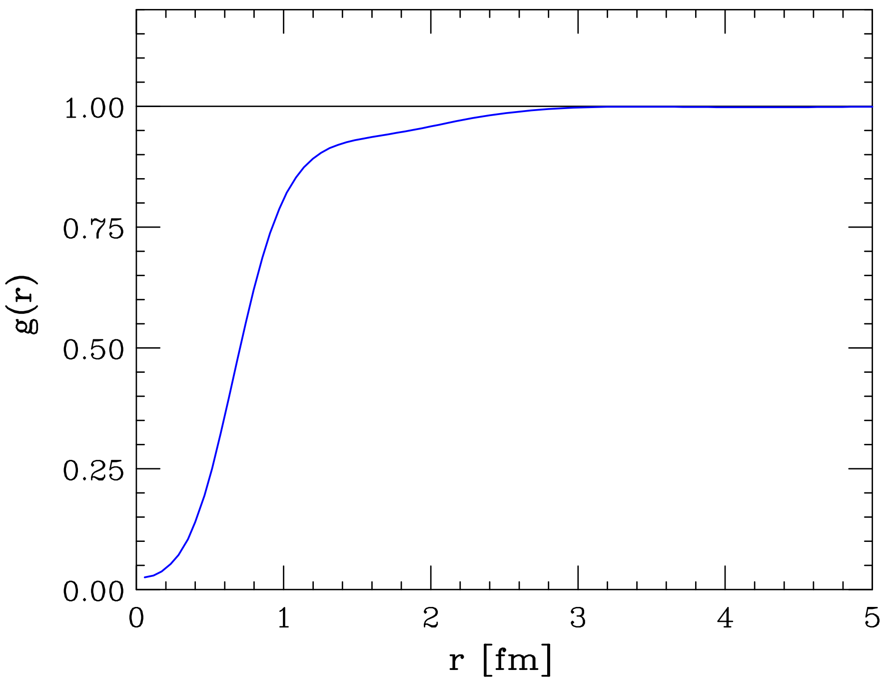
$$P_p(\mathbf{q}, \omega - E) = \frac{1}{\pi} \frac{\mathrm{Im}\, \widetilde{V}^0_{|\mathbf{q}|}}{\left[\omega - E - e^0_{|\mathbf{q}|} - \mathrm{Re}\, \widetilde{V}^0_{|\mathbf{q}|}\right]^2 + \left[\mathrm{Im}\, \widetilde{V}^0_{|\mathbf{q}|}\right]^2}$$

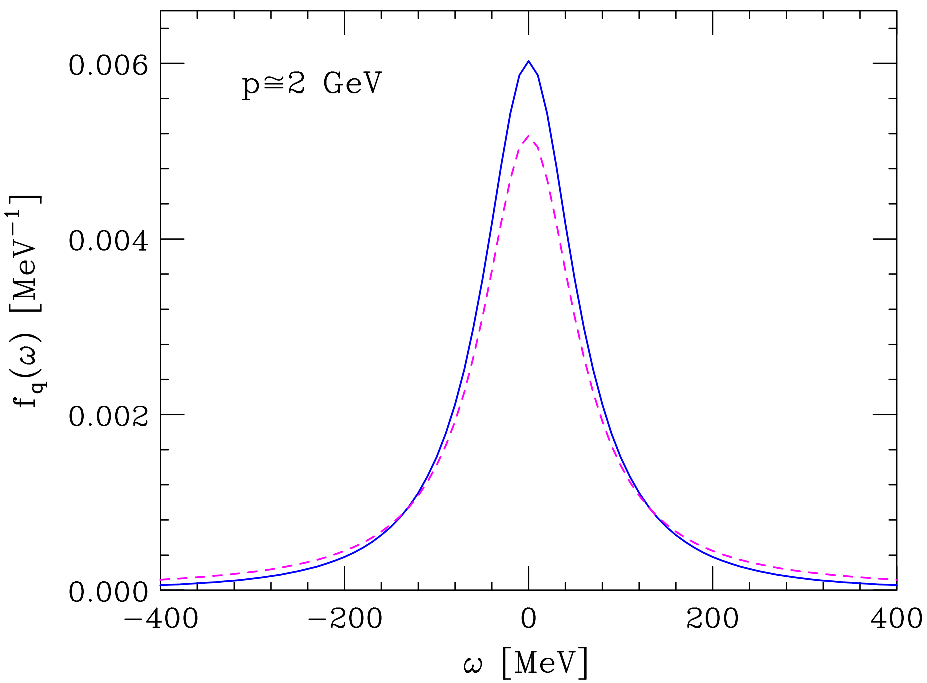
⋆ In presence of NN correlations inducing strong density fluctuations

$$\rho(\mathbf{r_1},\mathbf{r_2}) = \rho^2 g(|\mathbf{r_1}-\mathbf{r_2}|)$$

and (use the eikonal approximation)

$$\widetilde{V}_{|\mathbf{q}|}(z) = \widetilde{V}^0_{|\mathbf{q}|}\,\frac{1}{z}\int_0^z dz' g(z')$$



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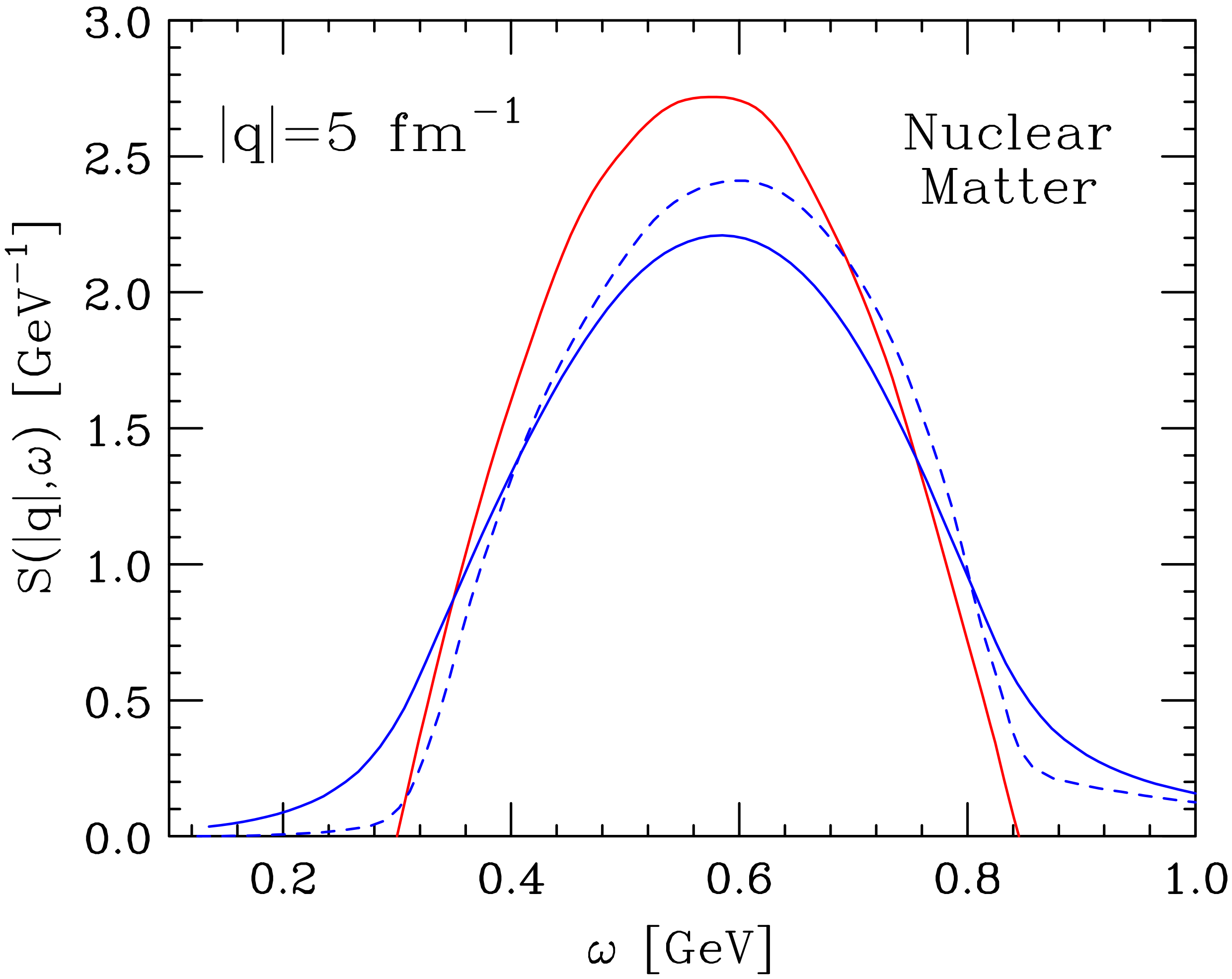

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- ⋆ In the impulse approximation regime the response can be conveniently expressed using the spectral function formalism
- ⋆ Theoretical calculations of the spectral function must include the contribution of correlations, which are known to be sizable
- ⋆ The hole-spectral function, describing initial state dynamics, can be obtained from nonrelativistic NMBT
- ⋆ The eikonal approximation provides a consistent framework suitable to carry out theoretical calculations of the particle spectral function in the regime of large ($\gtrsim 500$ MeV) momentum tansfer