# Green Function Formalism and Electroweak Nuclear Response

## Lecture 1

Omar Benhar

INFN and Department of Physics

Università “La Sapienza”, I-00185 Roma

# Outline

- ⋆ The paradigm of nuclear many-body theory (NMBT)

# Outline

- ⋆ The paradigm of nuclear many-body theory (NMBT)
  - ▷ The nucleon-nucleon (NN) interaction

# Outline

* The paradigm of nuclear many-body theory (NMBT)
    * The nucleon-nucleon (NN) interaction
    * Splendor and misery of the nuclear shell model

# Outline

- ⋆ The paradigm of nuclear many-body theory (NMBT)
  - ▷ The nucleon-nucleon (NN) interaction
  - ▷ Splendor and misery of the nuclear shell model
  - ▷ Many-body approaches based on *realistic* dynamical models

# Outline

- ⋆ The paradigm of nuclear many-body theory (NMBT)
  - ▷ The nucleon-nucleon (NN) interaction
  - ▷ Splendor and misery of the nuclear shell model
  - ▷ Many-body approaches based on *realistic* dynamical models
- ⋆ Many-body theory of the nuclear response

## Outline

- ⋆ The paradigm of nuclear many-body theory (NMBT)
  - ▷ The nucleon-nucleon (NN) interaction
  - ▷ Splendor and misery of the nuclear shell model
  - ▷ Many-body approaches based on *realistic* dynamical models
- ⋆ Many-body theory of the nuclear response
  - ▷ Formalism

## Outline

- ⋆ The paradigm of nuclear many-body theory (NMBT)
  - ▷ The nucleon-nucleon (NN) interaction
  - ▷ Splendor and misery of the nuclear shell model
  - ▷ Many-body approaches based on *realistic* dynamical models
- ⋆ Many-body theory of the nuclear response
  - ▷ Formalism
  - ▷ The low-energy (non relativistic) regime

# Outline

- ⋆ The paradigm of nuclear many-body theory (NMBT)
  - ▷ The nucleon-nucleon (NN) interaction
  - ▷ Splendor and misery of the nuclear shell model
  - ▷ Many-body approaches based on *realistic* dynamical models
- ⋆ Many-body theory of the nuclear response
  - ▷ Formalism
  - ▷ The low-energy (non relativistic) regime
  - ▷ Effects of short- and long-range nucleon-nucleon correlations

## The nucleon-nucleon (NN) interaction

- ⋆ The main features of the NN force can be inferred from nuclear systematics and from the properties of the two-nucleon systems (in both bound and scattering states).

## The nucleon-nucleon (NN) interaction

- ⋆ The main features of the NN force can be inferred from nuclear systematics and from the properties of the two-nucleon systems (in both bound and scattering states).
- ⋆ This empirical information suggests that the interaction potential $v_{NN}$ is

# The nucleon-nucleon (NN) interaction

⋆ The main features of the NN force can be inferred from nuclear systematics and from the properties of the two-nucleon systems (in both bound and scattering states).

⋆ This empirical information suggests that the interaction potential $v_{NN}$ is

  ▷ short ranged

## The nucleon-nucleon (NN) interaction

- The main features of the NN force can be inferred from nuclear systematics and from the properties of the two-nucleon systems (in both bound and scattering states).
- This empirical information suggests that the interaction potential $v_{NN}$ is
  - short ranged
  - strongly repulsive at short distance

# The nucleon-nucleon (NN) interaction

- ⋆ The main features of the NN force can be inferred from nuclear systematics and from the properties of the two-nucleon systems (in both bound and scattering states).
- ⋆ This empirical information suggests that the interaction potential $v_{NN}$ is
  - ▷ short ranged
  - ▷ strongly repulsive at short distance
  - ▷ strongly spin dependent

## The nucleon-nucleon (NN) interaction

- ⋆ The main features of the NN force can be inferred from nuclear systematics and from the properties of the two-nucleon systems (in both bound and scattering states).
- ⋆ This empirical information suggests that the interaction potential $v_{NN}$ is
  - ▷ short ranged
  - ▷ strongly repulsive at short distance
  - ▷ strongly spin dependent
  - ▷ non central

## The nucleon-nucleon (NN) interaction

- ⋆ The main features of the NN force can be inferred from nuclear systematics and from the properties of the two-nucleon systems (in both bound and scattering states).
- ⋆ This empirical information suggests that the interaction potential $v_{NN}$ is
  - ▷ short ranged
  - ▷ strongly repulsive at short distance
  - ▷ strongly spin dependent
  - ▷ non central
- ⋆ The analysis of deuteron and NN scattering at large angular momentum shows that one-pion-exchange (OPE) is the dominant interaction mechanism at large distance

⋆ Yukawa's OPE potential:

$$
\begin{aligned}
v_\pi &= \frac{g^2}{4m_N^2}\,(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla})\,\frac{\mathrm{e}^{-m_\pi r}}{r} \\
&= \frac{g^2}{(4\pi)^2}\,\frac{m_\pi^3}{4m^2}\,\frac{1}{3}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left\{\left[(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+S_{12}\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\right]\frac{\mathrm{e}^{-x}}{x}\right. \\
&\qquad\left. -\,\frac{4\pi}{m_\pi^3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\delta^{(3)}(\mathbf{r})\right\}\ ,
\end{aligned}
$$

where $g^2/(4\pi)=14$, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, $x=m_\pi r$ and

$$
S_{12}=\frac{3}{r^2}(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\ ,
$$

is reminiscent of the operator describing the noncentral interaction between two magnetic dipoles.

⋆ Phenomenological potentials describing the *full* NN interaction are generally cast in the form

$$v = v_S + v_I + \widetilde{v}_\pi \tag{1}$$

where $\widetilde{v}_\pi$ is the OPE potential, stripped of the $\delta$-function contribution

- ⋆ Phenomenological potentials describing the *full* NN interaction are generally cast in the form

$$v = v_S + v_I + \widetilde{v}_\pi \tag{1}$$

where $\widetilde{v}_\pi$ is the OPE potential, stripped of the $\delta$-function contribution

- ⋆ State-of-the-art NN potential models, yielding an accurate description of NN scattering data, use parametrized $v_S$ and $v_I$, including momentum-dependent and charge-symmetry breaking terms. The widely used ANL $v_{18}$ potential is written in the form

$$v_{12} = \sum_{p=1,18} v^p(r) O^p_{12}$$

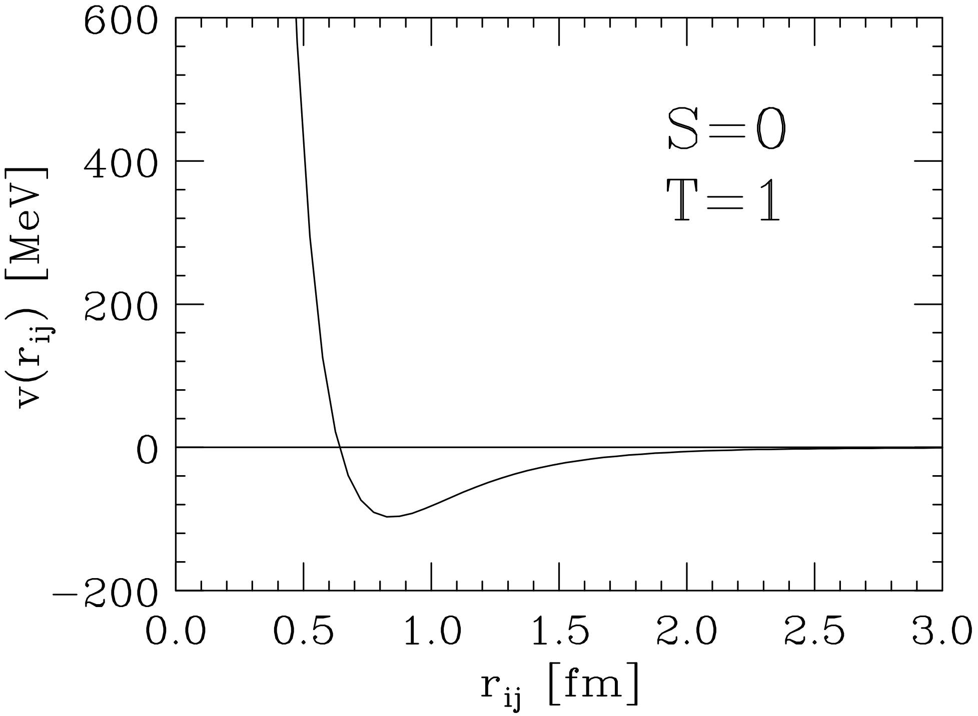
$$O^p_{12} = [1, \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, S_{12}, \mathbf{L} \cdot \mathbf{S}, \mathbf{L}^2, \mathbf{L}^2(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2), (\mathbf{L} \cdot \mathbf{S})^2] \otimes [1, \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2] \ ,$$

$$[1, \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, S_{12}] \otimes T_{12} \ , \ \text{and} \ (\tau_{z1} + \tau_{z2})$$

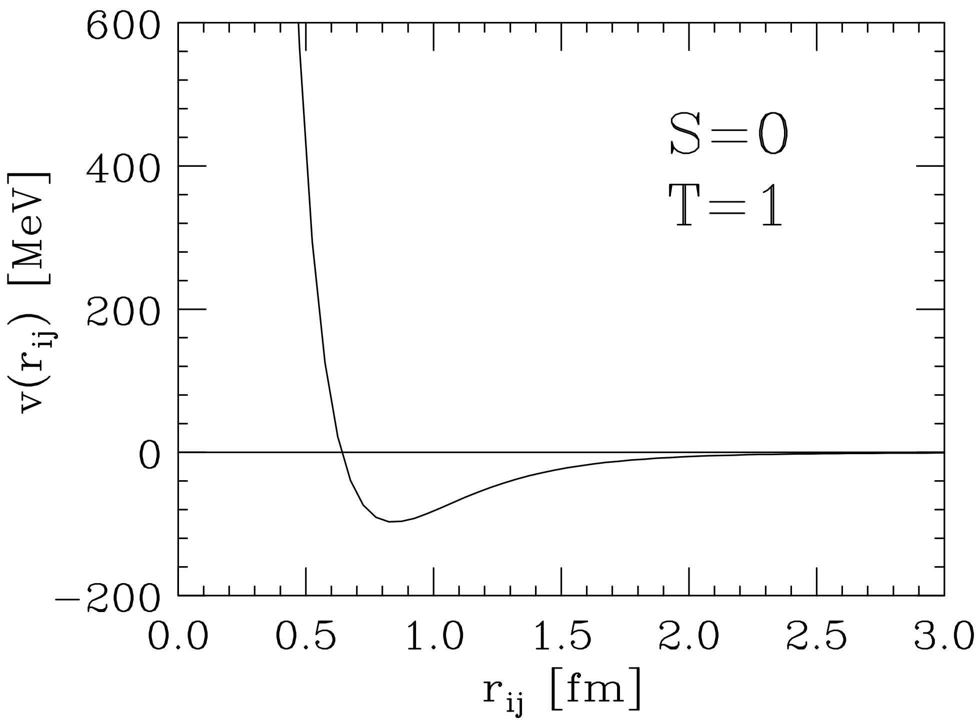
$$T_{12} = \frac{3}{r^2}(\boldsymbol{\tau}_1 \cdot \mathbf{r})(\boldsymbol{\tau}_2 \cdot \mathbf{r}) - (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$$

- Radial dependence of the potential describing the interaction between two nucleons in the state of relative angular momentum $\ell = 0$, and total spin and isospin $S = 0$ and $T = 1$

⋆ Radial dependence of the potential describing the interaction between two nucleons in the state of relative angular momentum $\ell = 0$, and total spin and isospin $S = 0$ and $T = 1$



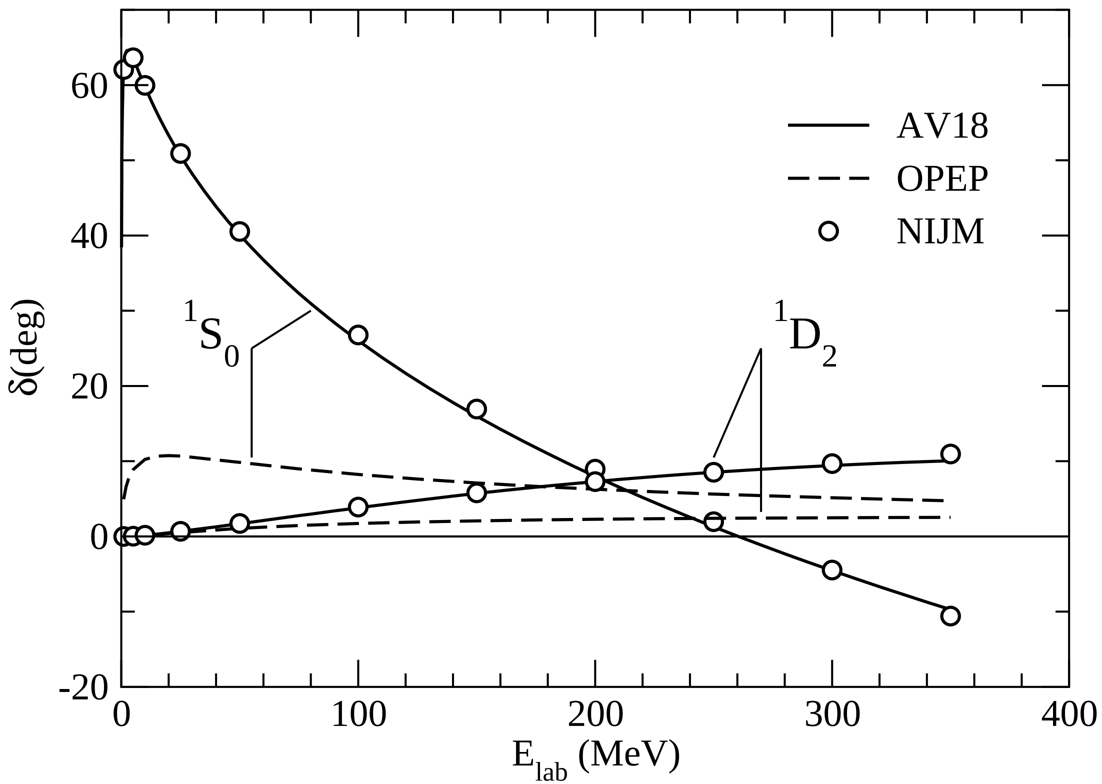
⋆ Radial dependence of the potential describing the interaction between two nucleons in the state of relative angular momentum $\ell = 0$, and total spin and isospin $S = 0$ and $T = 1$



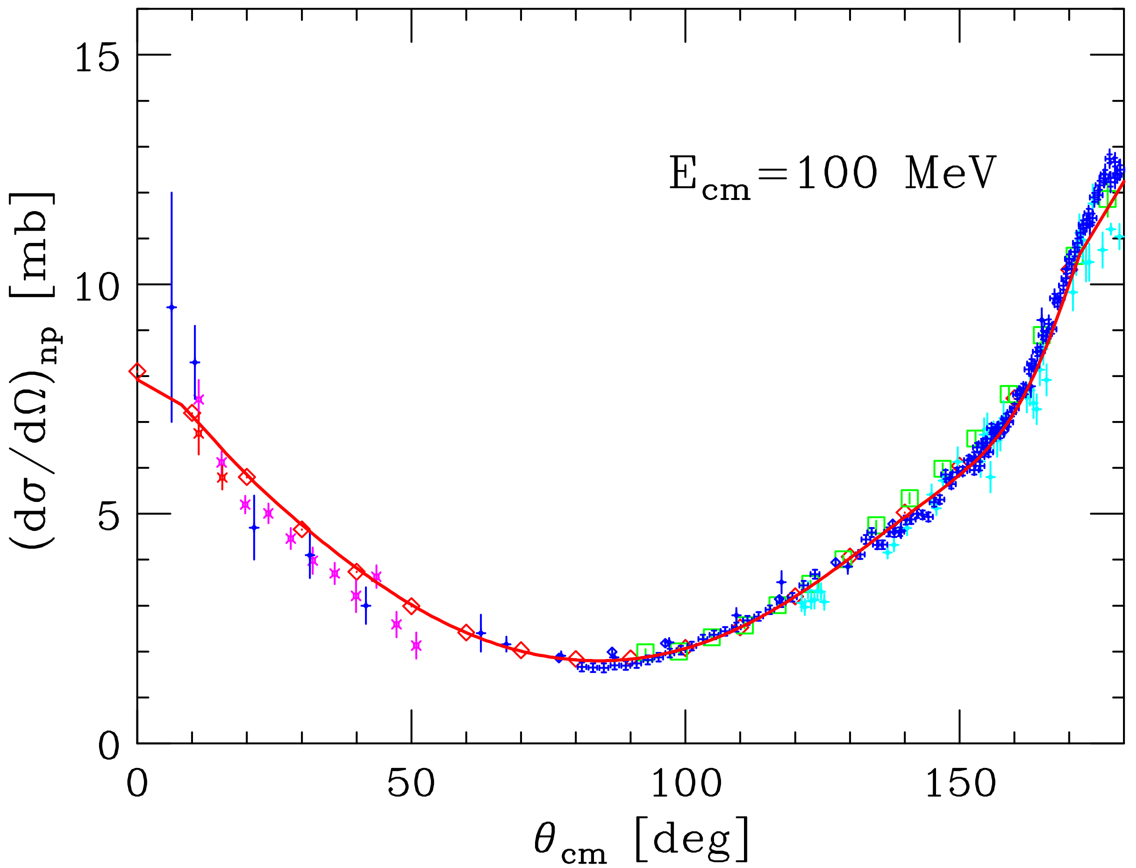

⋆ Note: the NN potental looks very similar to the interaction potential of a van der Waals liquid

⋆ The ANL $v_{18}$ potential provides an excellent fit of the ($\sim$ 4000) phase shifts in the Nijmegen $pp$ and $np$ scattering data base, low-energy scattering parameters, and deuteron binding energy.

⋆ The ANL $v_{18}$ potential provides an excellent fit of the ($\sim$ 4000) phase shifts in the Nijmegen $pp$ and $np$ scattering data base, low-energy scattering parameters, and deuteron binding energy.



⋆ Proton-neutron scattering x-section at center of mass energy $E_{cm} = 100$ MeV calculated using the ANL $v_{18}$ potential.



# The mean field approximation (shell model)

- ⋆ The nuclear shell model is based on the assumption that the interaction terms appearing in the hamiltonian can be replaced by a mean field according to

$$\sum_{j>i} v_{ij} + \ldots \rightarrow \sum_i U_i$$

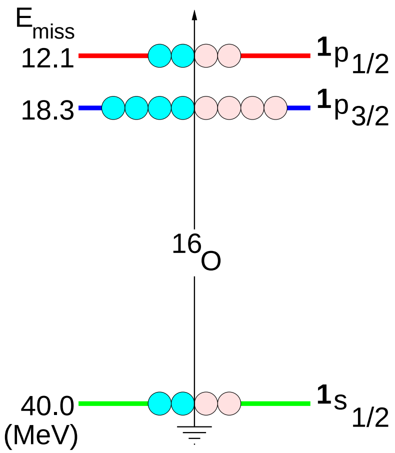
- ⋆ Within this picture the many body Schrödinger equation reduces to a single particle problem

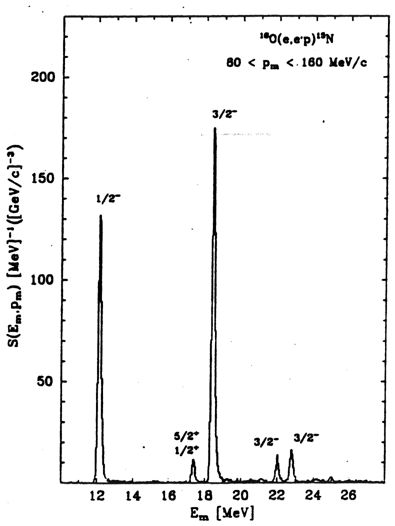
$$\left( \frac{\mathbf{p}_i^2}{2m} + U_i \right) \phi_{\alpha_i}(i) = \epsilon_{\alpha_i} \phi_{\alpha_i}(i) \quad , \quad \Phi_A = \mathcal{A} \prod_{\alpha_i \epsilon \{F\}} \phi_{\alpha_i}(i) \ ,$$

$\phi_{\alpha_i}(i)$ and $\epsilon_{\alpha_i}$ being the wave function and energy of the shell model state $\alpha_i$, respectively

⋆ There is ample empirical evidence that the shell model is not the whole story !

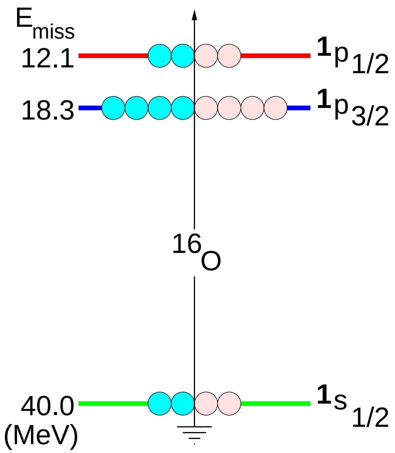
⋆ There is ample empirical evidence that the shell model is not the whole story !

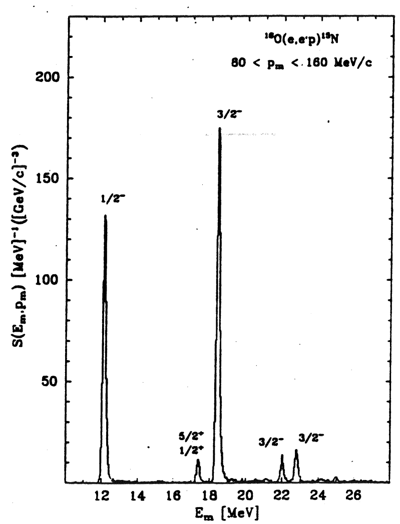


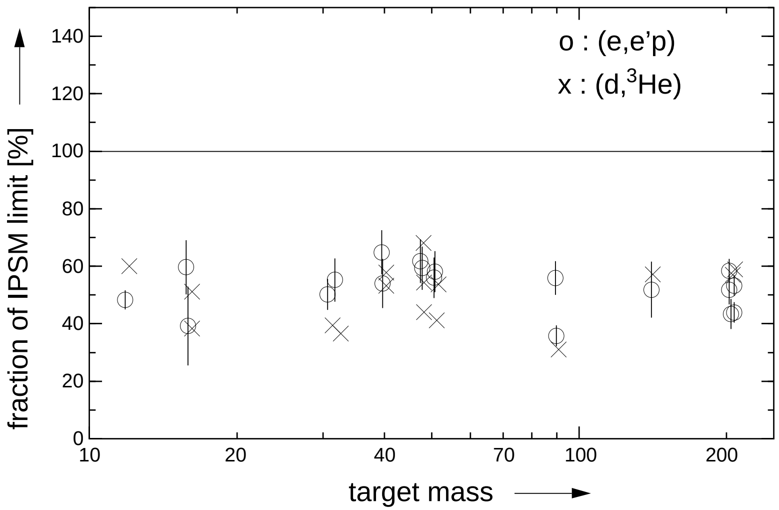




▷ Spectral lines corresponding to shell model orbitals clearly seen in high resolution $(e, e'p)$ experiments

★ There is ample empirical evidence that the shell model is not the whole story !




▷ Spectral lines corresponding to shell model orbitals clearly seen in high resolution $(e, e'p)$ experiments

▷ The measured strengths of the peaks corresponding to shell model states are significantly less than unity ($\lesssim 0.6$ for valence states)

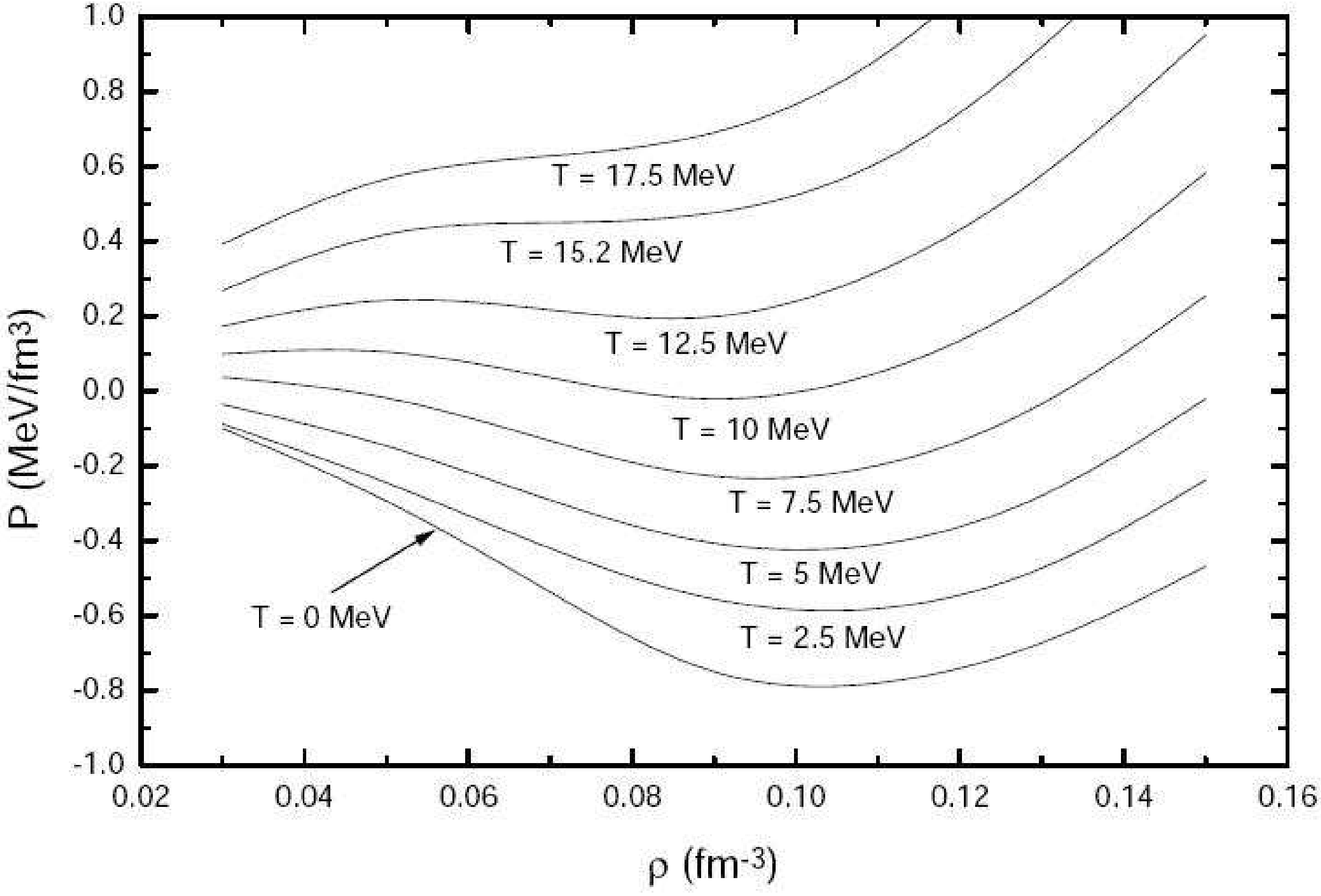
## Effects of nucleon-nucleon correlations

- ⋆ The results of nucleon knock out experiments show that a significant fraction of the target nucleons *do not behave as independent particles*, thus providing a clear signature of *correlation* effects
- ⋆ The simplest example of correlated system is the van der Waals liquid, described by the equation of state

$$(P - an^2)(V - nb) = nT$$

- ⋆ The van der Waals interaction potential is very similar to the NN potential
- ⋆ The failure of the mean field approximation to describe the van der Waals liquid (see, e.g., Kadanoff & Baym *Quantum Statistical Mechanics*) suggests that correlations also play a critical role in nuclei, and cannot be disregarded

⋆ Nuclear matter isotherms look very much the same as those of the van der Waals fluid.



## Many-body approaches based on realistic dynamical models

- ⋆ Bottom line: decouple the uncertainty associated with dynamical models from the approximations implied in many-body calculations

## Many-body approaches based on realistic dynamical models

- ⋆ Bottom line: decouple the uncertainty associated with dynamical models from the approximations implied in many-body calculations

- ⋆ Dynamics determined from the properties of two- and three-nucleon systems (exactly solvable)

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m} + \sum_{j>i} v_{ij} + \sum_{k>j>i} V_{ijk}$$

where $v_{ij}$ is a realistic NN potential (e.g. the ANL $v_{18}$) and $V_{ijk}$ is needed to reproduce the energies of the the three-nucleon systems

$$\langle V_{ijk} \rangle \ll \langle v_{ij} \rangle$$

# Many-body approaches based on realistic dynamical models

- ⋆ Bottom line: decouple the uncertainty associated with dynamical models from the approximations implied in many-body calculations

- ⋆ Dynamics determined from the properties of two- and three-nucleon systems (exactly solvable)

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m} + \sum_{j>i} v_{ij} + \sum_{k>j>i} V_{ijk}$$

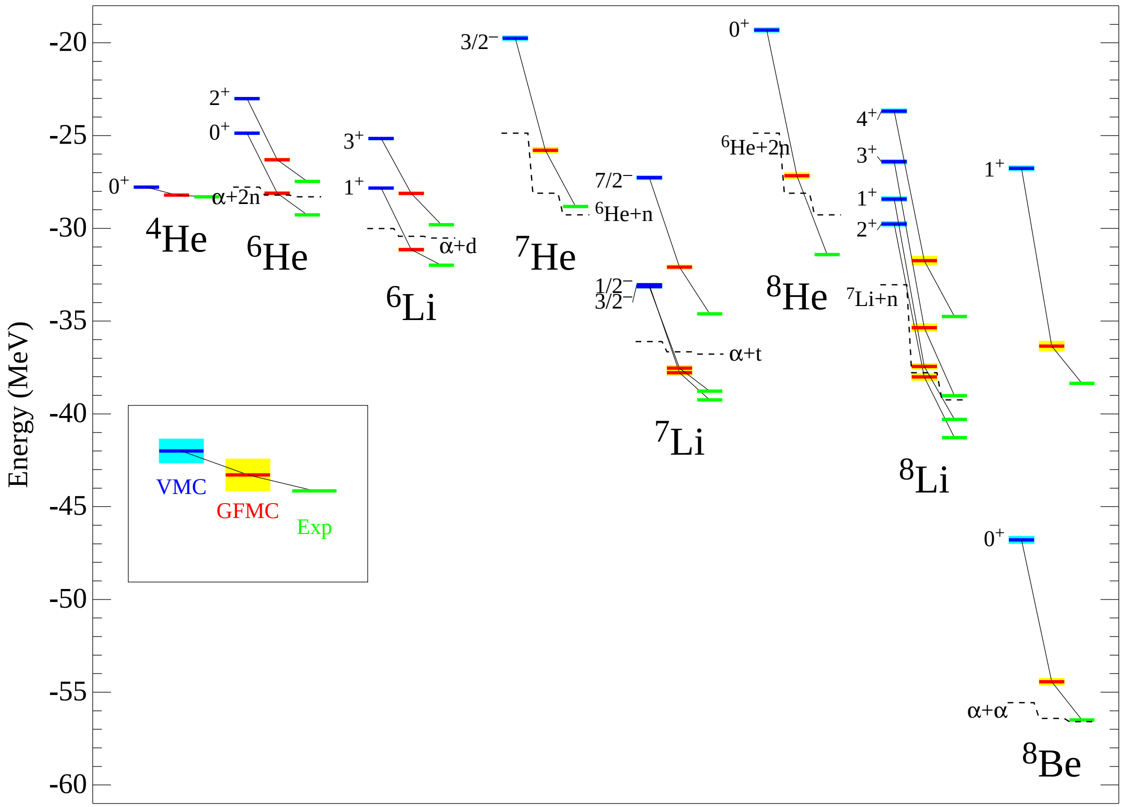
  where $v_{ij}$ is a realistic NN potential (e.g. the ANL $v_{18}$) and $V_{ijk}$ is needed to reproduce the energies of the the three-nucleon systems

$$\langle V_{ijk} \rangle \ll \langle v_{ij} \rangle$$

- ⋆ Calculations of nuclear observables *do not* involve any adjustable parameters

⋆ Ground and low-lying excited states of nuclei with $A \leq 8$. No approximations involved in the solution of the Schrödinger equation

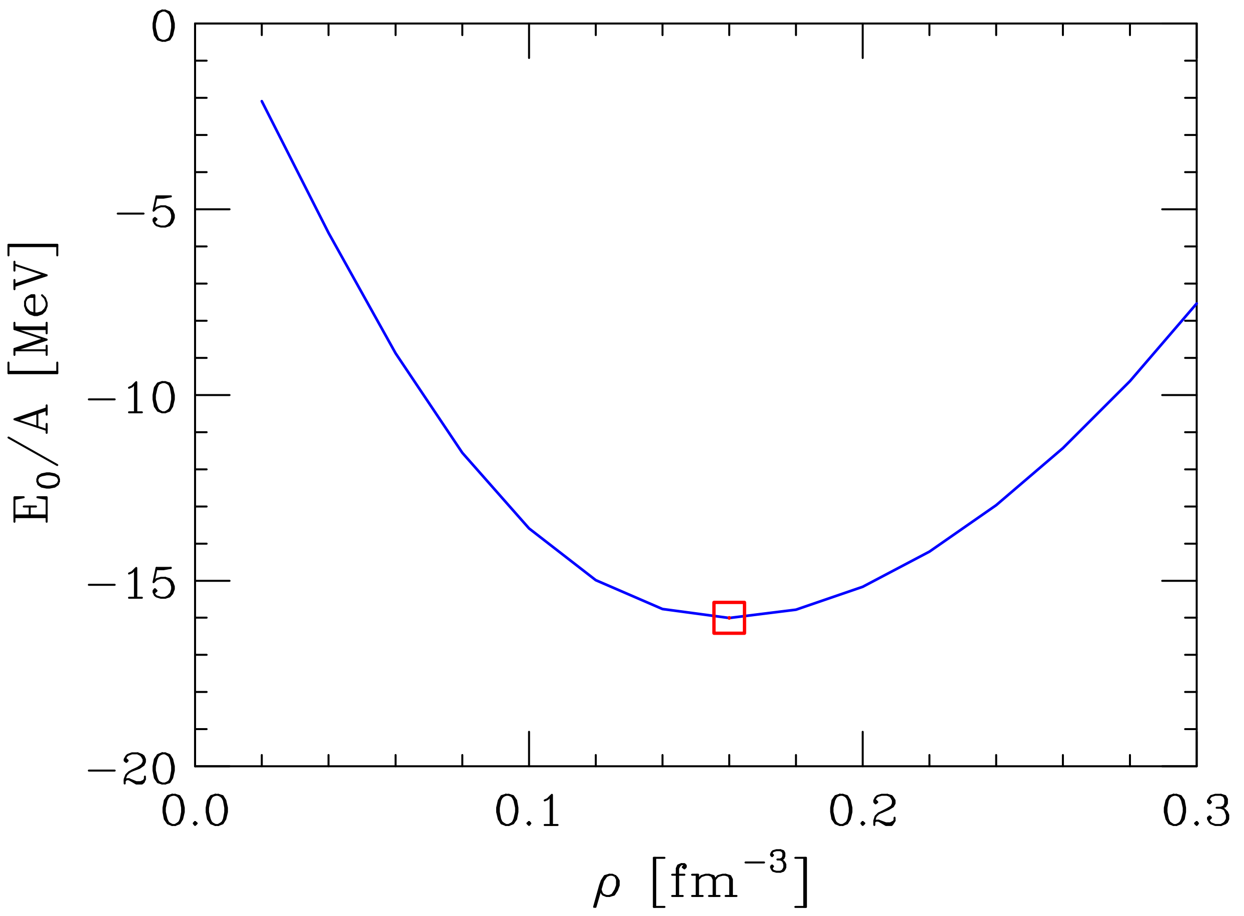
$$H|n\rangle = E_n|n\rangle$$




$$\Delta = \frac{|E_n^{GFMC} - E_n^{exp}|}{E_n^{exp}} \lesssim 5\%$$

⋆ Note: these calculations are now doable for nuclei with $A \leq 12$.

⋆ Exploiting translation invariance, accurate calculations can also be carried out for uniform nuclear matter in the limit $A \to \infty$



## The nuclear response within NMBT

- ⋆ Consider scattering of a scalar probe, for simplicity

$$\frac{d\sigma}{d\Omega d\omega} \propto S(\mathbf{q},\omega) = \sum_n \langle 0|\rho^\dagger_{\mathbf{q}}|n\rangle\langle n|\rho_{\mathbf{q}}|0\rangle\delta(E_0+\omega-E_n)$$

$$\rho_{\mathbf{q}} = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}}$$

$$H|0\rangle = E_0|0\rangle \quad , \quad H|n\rangle = E_n|n\rangle$$

## The nuclear response within NMBT

- ⋆ Consider scattering of a scalar probe, for simplicity

$$\frac{d\sigma}{d\Omega d\omega} \propto S(\mathbf{q},\omega) = \sum_n \langle 0|\rho^\dagger_{\mathbf{q}}|n\rangle\langle n|\rho_{\mathbf{q}}|0\rangle \delta(E_0 + \omega - E_n)$$

$$\rho_{\mathbf{q}} = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}}$$

$$H|0\rangle = E_0|0\rangle \quad , \quad H|n\rangle = E_n|n\rangle$$

- ⋆ At $|\mathbf{q}| \lesssim 500$ MeV, exact calculations are feasible for $A \leq 4$ using integral transform techniques

## The nuclear response within NMBT

- ⋆ Consider scattering of a scalar probe, for simplicity

$$\frac{d\sigma}{d\Omega d\omega} \propto S(\mathbf{q},\omega) = \sum_n \langle 0|\rho^\dagger_{\mathbf{q}}|n\rangle\langle n|\rho_{\mathbf{q}}|0\rangle \delta(E_0+\omega-E_n)$$

$$\rho_{\mathbf{q}} = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}}$$

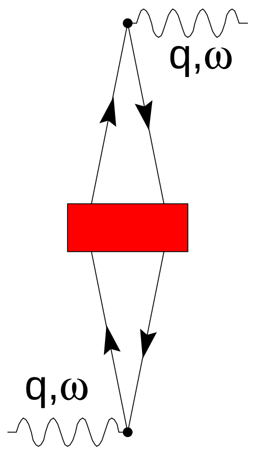
$$H|0\rangle = E_0|0\rangle \quad , \quad H|n\rangle = E_n|n\rangle$$

- ⋆ At $|\mathbf{q}| \lesssim 500\ \mathrm{MeV}$, exact calculations are feasible for $A \leq 4$ using integral transform techniques
- ⋆ Accurate results for uniform nuclear matter have been also obtained (exploiting again translation invariance) from Correlated Basis Function (CBF) perturbation theory (to be discussed later)

⋆ Rewrite $S(\mathbf{q},\omega)$ in the form

$$S(\mathbf{q},\omega) = \sum_n \Big| \sum_k \langle n | a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} | 0 \rangle \Big|^2 \, \delta(\omega + E_0 - E_n)$$

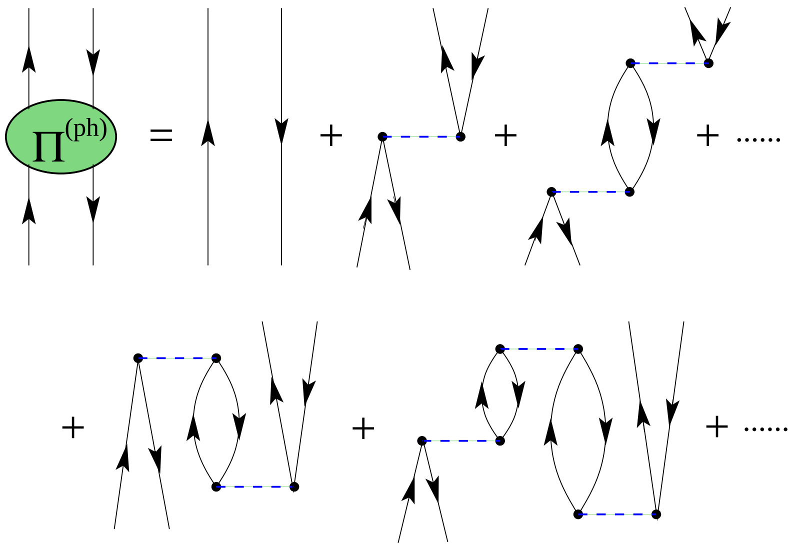
$$= \int \frac{dt}{2\pi} \, \mathrm{e}^{i(\omega+E_0)t} \sum_{p,k} \langle 0 | a_{\mathbf{p}+\mathbf{q}} a^\dagger_{\mathbf{p}} \, \mathrm{e}^{-iHt} \, a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} | 0 \rangle$$

$=$ 


⋆ $S(\mathbf{q},\omega)$ can be expressed in terms of interactions and propagators (Green functions) of nucleons in particle and hole states

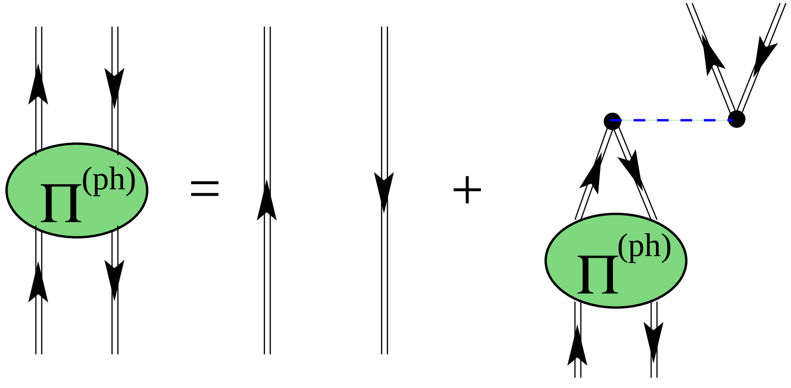
⋆ The polarization propagator $\Pi(\mathbf{q}, \omega)$ can be computed in a variety of approximation schemes using a diagrammatic expansion

RPA

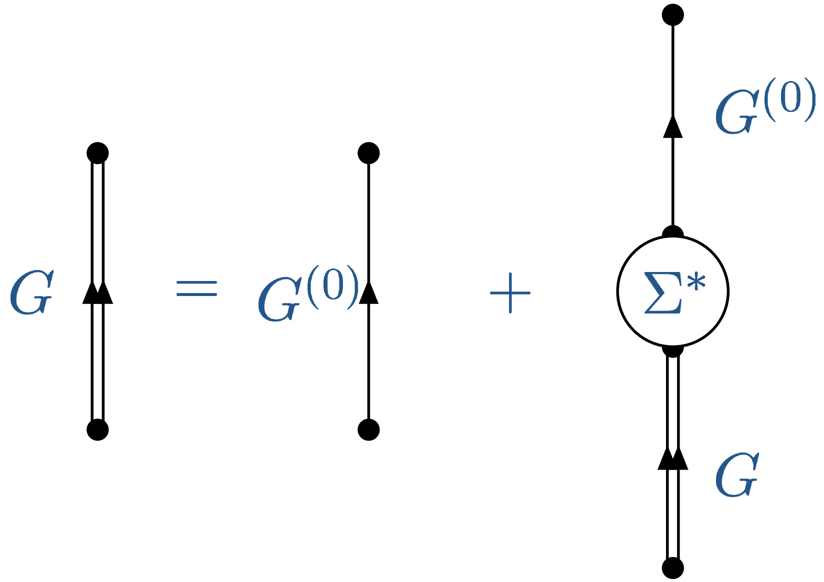


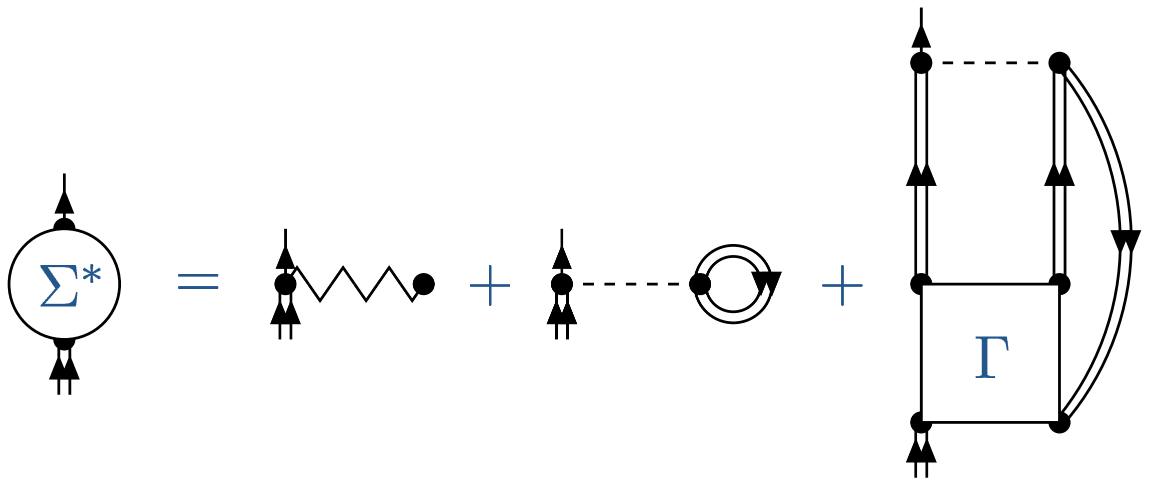

*dressing* all particle and hole lines leads to

DRPA



# Introducing the Green function



## Effects of NN interactions on the nuclear response

* In the absence of correlations, the only possible final states are one particle-one hole states

## Effects of NN interactions on the nuclear response

- ⋆ In the absence of correlations, the only possible final states are one particle-one hole states
- ⋆ For example, according to the Fermi gas model

$$M_n = \langle n | \sum_{\mathbf{k}} a^{\dagger}_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} | 0 \rangle \to M_k = 1 \times \theta(k_F - |\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}| - k_F)$$

$$S(\mathbf{q}, \omega) = \sum_{\mathbf{k}} |M_k|^2 \delta(\omega + e_0(\mathbf{k}) - e_0(\mathbf{k}+\mathbf{q}))$$

$$e_0(\mathbf{k}) = \frac{\mathbf{k}^2}{2m}$$

## Effects of NN interactions on the nuclear response

- ⋆ In the absence of correlations, the only possible final states are one particle-one hole states
- ⋆ For example, according to the Fermi gas model

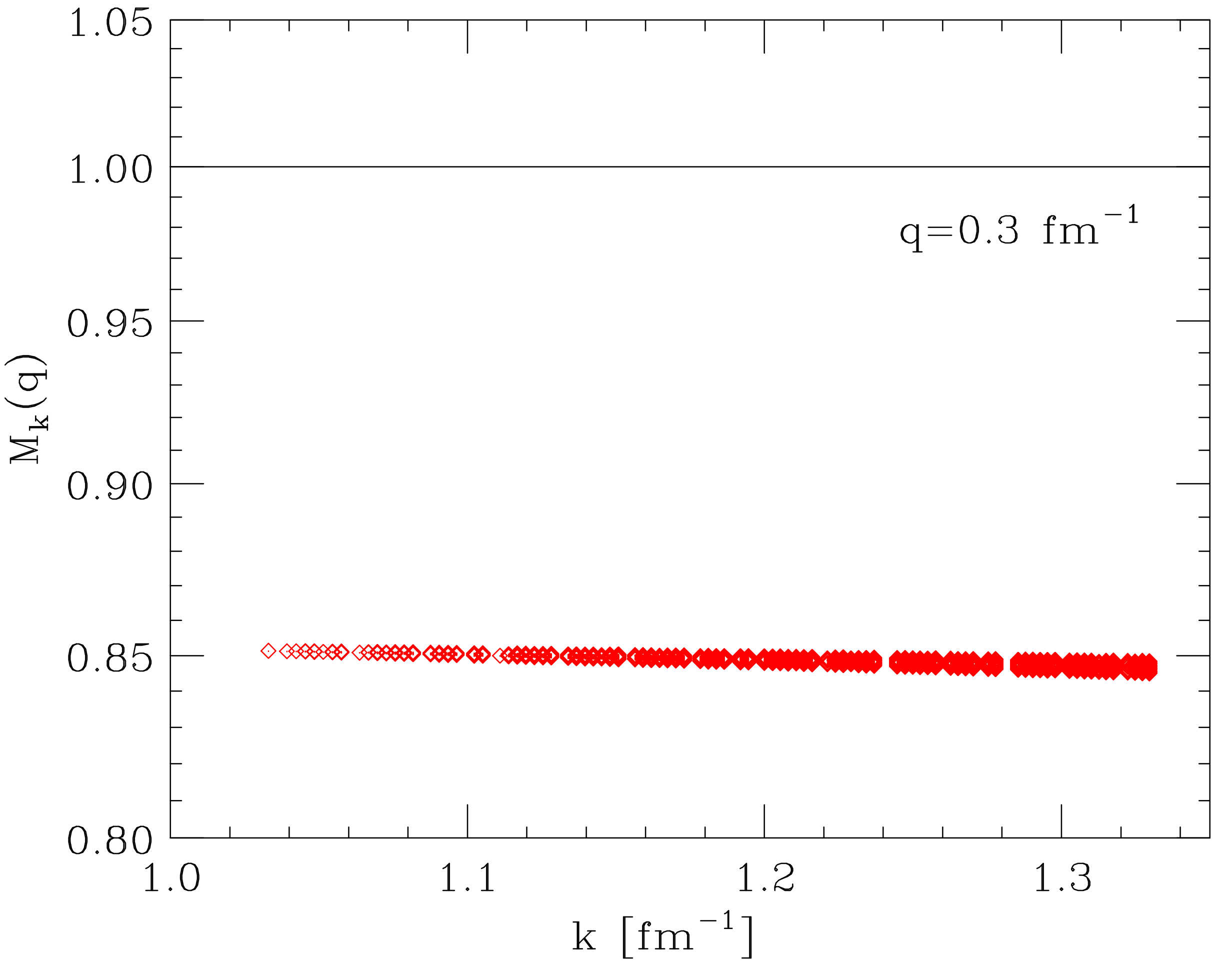
$$M_n = \langle n | \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} | 0 \rangle \to M_k = 1 \times \theta(k_F - |\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}| - k_F)$$

$$S(\mathbf{q}, \omega) = \sum_{\mathbf{k}} |M_k|^2 \delta(\omega + e_0(\mathbf{k}) - e_0(\mathbf{k}+\mathbf{q}))$$

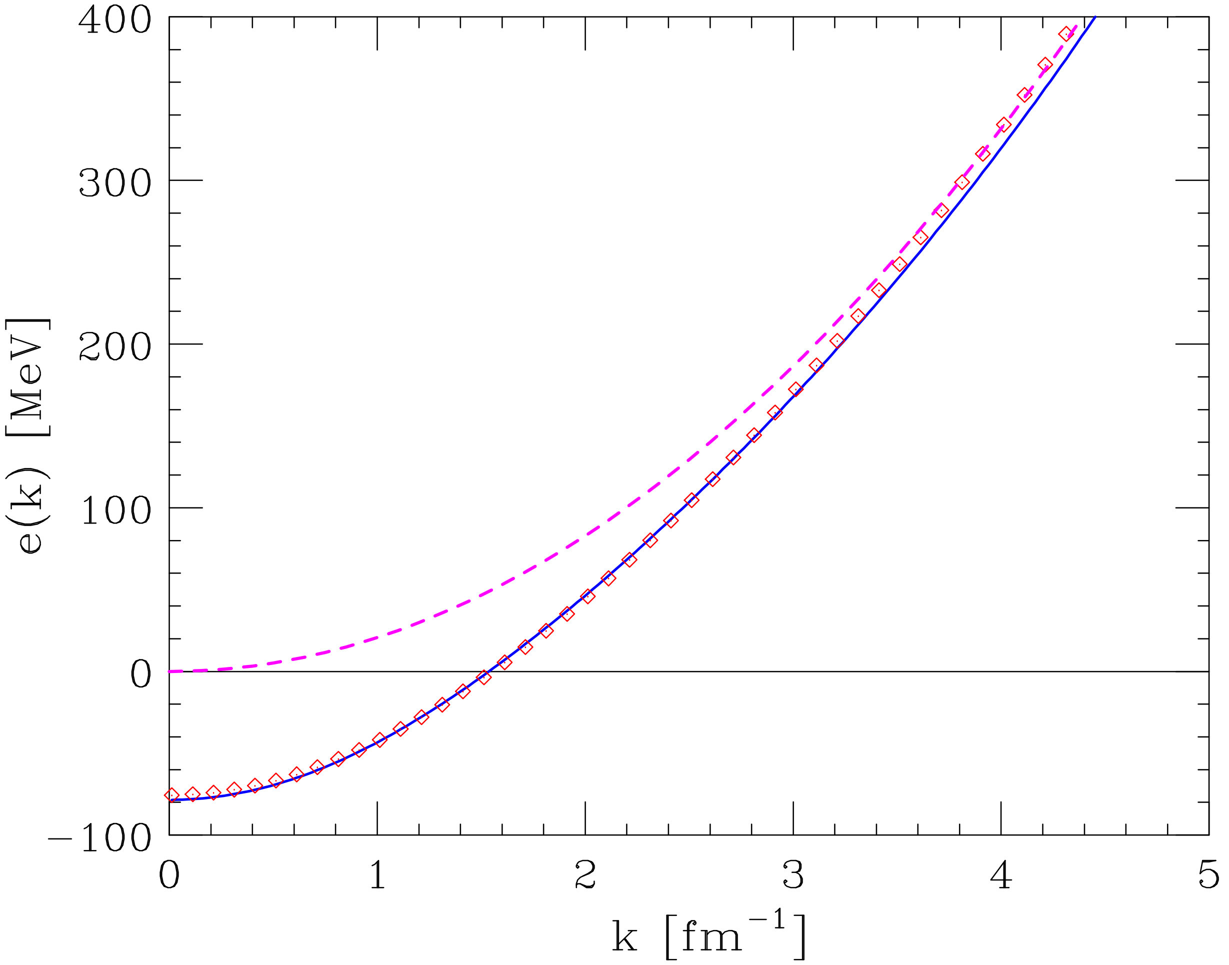
$$e_0(\mathbf{k}) = \frac{\mathbf{k}^2}{2m}$$

- ⋆ Inclusion of interactions leads to a quenching of the transition matrix elements $M_k$ and to a modification of the single particle spectrum $e_0(\mathbf{k})$

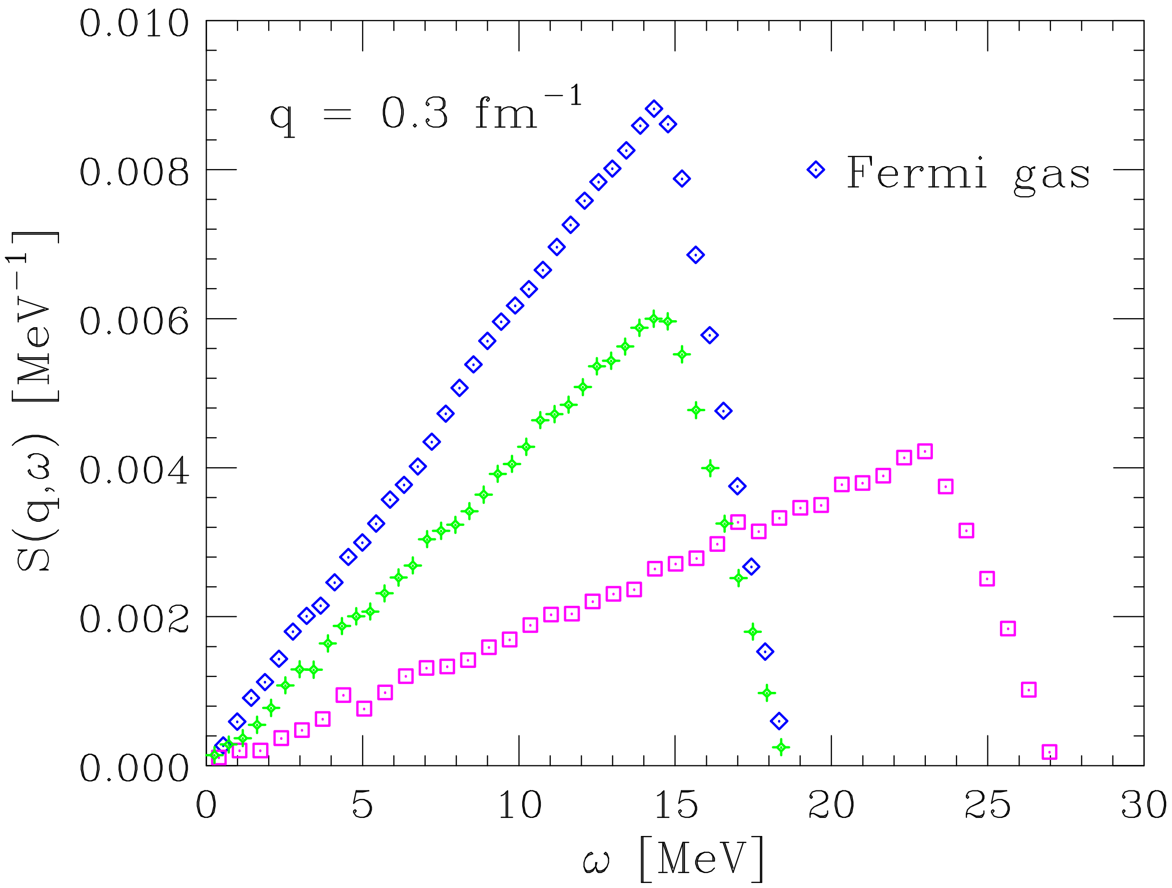
⋆ The quenching of the transition matrix elements to one particle-one hole states is due to short range correlations



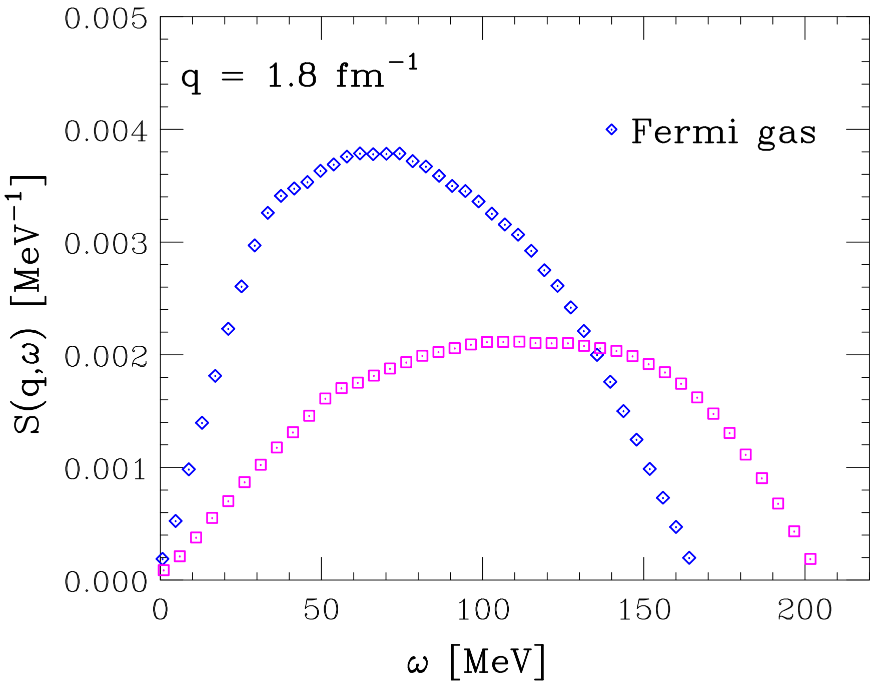
⋆ Deviation from the kinetic energy spectrum due to interaction effects included in the mean field

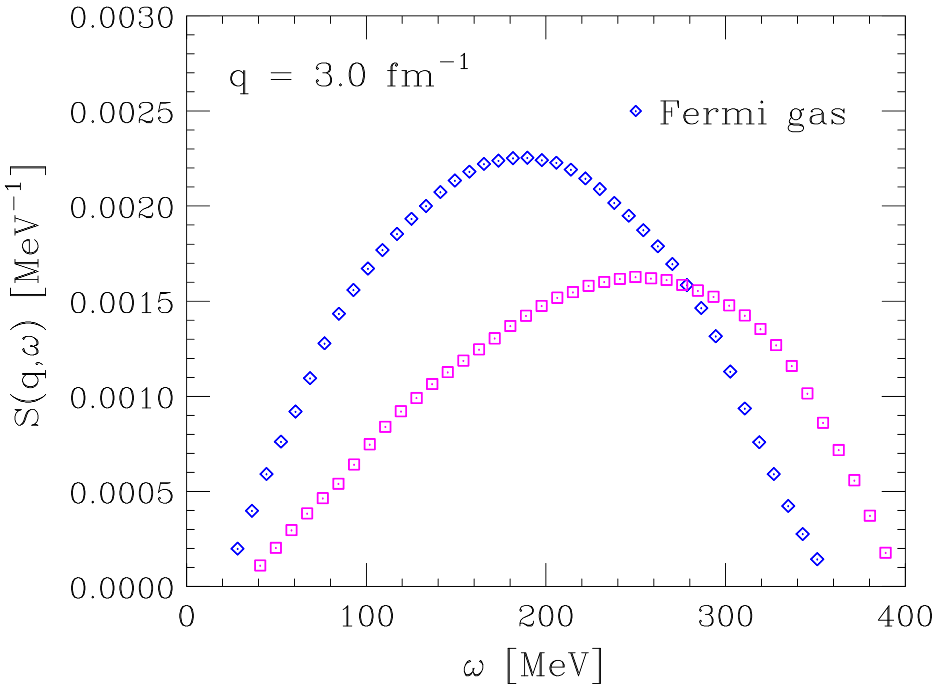


# Effects of NN interactions on the nuclear response



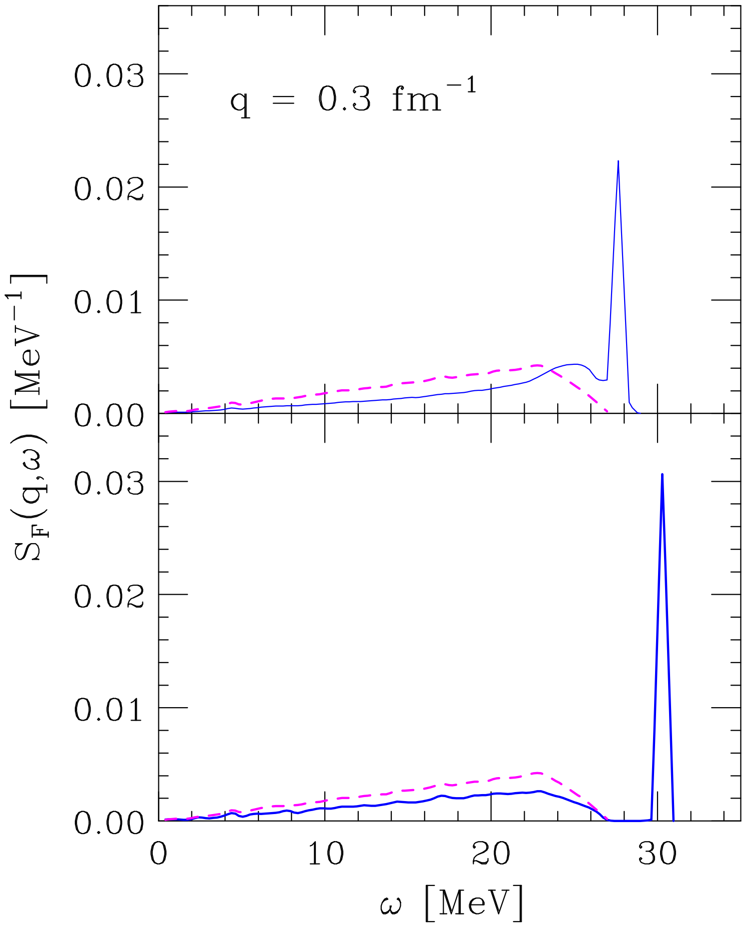
# Effects of NN interactions on the nuclear response (continued)



## Effects of long range correlations on $S(\mathbf{q}, \omega)$ @ low $|\mathbf{q}|$

⋆ Transitions between particle-hole states induce long range correlations leading to the excitation of a collective mode (reminiscent of the plasma oscillation) at low momentum transfer



## Summary

- ⋆ Nuclear many-body theory (NMBT) provides a consistent framework to carry out calculations of the nuclear response based on *realistic* dynamical models

## Summary

- Nuclear many-body theory (NMBT) provides a consistent framework to carry out calculations of the nuclear response based on *realistic* dynamical models
- Interaction effects neglected in the naive FG model are large

## Summary

- ⋆ Nuclear many-body theory (NMBT) provides a consistent framework to carry out calculations of the nuclear response based on *realistic* dynamical models
- ⋆ Interaction effects neglected in the naive FG model are large
- ⋆ Correlations (both short- and long-ranged) are important (indeed dominant in some kinematical regions)

## Summary

- ⋆ Nuclear many-body theory (NMBT) provides a consistent framework to carry out calculations of the nuclear response based on *realistic* dynamical models
- ⋆ Interaction effects neglected in the naive FG model are large
- ⋆ Correlations (both short- and long-ranged) are important (indeed dominant in some kinematical regions)
- ⋆ The nonrelativistic formalism is applicable at $|\mathbf{q}| \lesssim 500$ MeV. Its extension to higher energies requires the development of a suitable approximation scheme, to be discussed in the next Lecture.