The Standard Model

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1 Consistuents of the Standard Model

The Standard Model (SM) of particle physics is a theory describing all known types of elementary particles. This means all three generations of quarks and leptons, which form the matter, as well as particles mediating their interactions. This excludes the gravitational interaction, which remains a mystery from the point of particle physics.

Table 1: Leptons and quarks in the Standard Model.

Generation:	1		2		3	
Quarks	u (up)	m= 2.5$[MeV]$ Q=$\frac{2}{3}$ s=$\frac{1}{2}$	c (charm)	m= 1.29$[GeV]$ Q=$\frac{2}{3}$ s=$\frac{1}{2}$	t (top)	m= 172.9$[GeV]$ Q=$\frac{2}{3}$ s=$\frac{1}{2}$
	d (down)	m= 5$[MeV]$ Q=$-\frac{1}{3}$ s=$\frac{1}{2}$	s (strange)	m= 100$[MeV]$ Q=$-\frac{1}{3}$ s=$\frac{1}{2}$	b (bottom)	m= 4.19$[GeV]$ Q=$-\frac{1}{3}$ s=$\frac{1}{2}$
Leptons	e (electron)	m= 0.511$[MeV]$ Q=-1 s=$\frac{1}{2}$	μ (muon)	m= 105.66$[MeV]$ Q=-1 s=$\frac{1}{2}$	τ (tau)	m= 1.777$[GeV]$ Q=-1 s=$\frac{1}{2}$
	ν_e (electron neutrino)	m$<\approx$ 2$[eV]$ Q=0 s=$\frac{1}{2}$	ν_μ (muon neutrino)	m$<\approx$ 2$[eV]$ Q=0 s=$\frac{1}{2}$	ν_τ (tau neutrino)	m$<\approx$ 2$[eV]$ Q=0 s=$\frac{1}{2}$

In the table 1 we have listed the three generations of spin-1/2 paricles, which build matter. They are called leptons and quarks. Only in the first generation there are particles, which form our everyday surroundings: the u and d quarks form neutrons and protons, together with the electrons they are constituents of atoms. All of these particles have some common quantum numbers, like the electric charge Q (in the units of electron charge) of up, charm and top quarks are the same, as well as charges of the electron, muon and tau. But their masses are totall different. The first generation quarks and charged leptons are lighter, than the ones from the second and the ones from the seond are lighter than the ones from the third. Muon is about two hundred times heavier, than the electron, and the taon is about sixteen times heavier, than muon. The same trend can be observed among quarks: the up quark is rather light, it is supposed to have a free mass of around 5-6 electron masses, but the top quark is as heavy as the whole gold nucleus. Of course there are no such things, as "free quarks", but experimentalists can give us some estimates. The

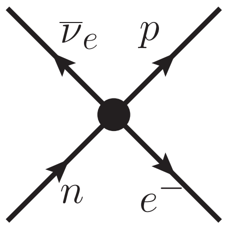

Figure 1: Neutron β-decay before the discovery of quarks and introduction of $W^{\pm}$ bosons.

only exception are masses of the neutrinos, which are unknown until today. There are only some rough constraints, which place the possible mass range below couple of eV. These neutral Fermions are exceptionally light. Compared to the charged ones and to quarks they can be treated as "massless" particles. All of the above listed matter constituents have their antiparticles with opposite electric charge. The only exception is again the neutrino, whose antiparticle must be neutral and we do not know yet, whether the neutrino is its own antiparticle (Majorana/Dirac type) or not. The heavy particles tend to decay very quickly into their lighter relatives. This is the reason, why in the everyday matter only first generation contributes.
It is also worthy to notice, that all the detected neutrinos are left-handed *chirality*[1] particles. No right-handed neutrino interactions have ever been recorded! This has led, historically, to a first Hamiltonian containing parity breaking terms. At that time most of the known weak proceses were simple β-decays of neutrons into a proton, electron and electron antineutrino (see fig. 1). Quarks and weak interaction bosons were unknown. Thus Feynman and Gell-Mann ([1]) have proposed a Hamiltonian connecting the left-handed components of fermionic currents (neutron-proton and electron-neutrino):

$$\begin{aligned} \mathcal{H} &= \frac{G_F}{\sqrt{2}} 4\overline{p}_L\gamma_\mu n_L\overline{e}_L\gamma^\mu\nu_L + h.c. \qquad (1.1) \\ &= \frac{G_F}{\sqrt{2}}\overline{p}\gamma_\mu(1-\gamma_5)n\overline{e}\gamma^\mu(1-\gamma_5)\nu + h.c. \end{aligned}$$

with G_F being the Fermi coupling constant and $\frac{1-\gamma_5}{2}$ - the left-handed helicity projection operator. The above Hamiltonian, although succesive at low interaction energies, has a few drawbacks. This type of interaction is nonrenormalizable, so one can consider loop-level processes, like the one in fig. 1. And since the electromagnetic interaction is carried by a photon, there must be a particle responsible for the weak one as well. Here we have no information about it.

Table 2 contains the interaction carriers known in SM. There are three types of them: strong interaction of quarks, electromagnetic interaction of charged particles, and weak interaction concerning all spin-1/2 particles. Notice, that neutrinos interact here only weakly. All of these interacitions are mediated by spin-1 bosons. The strong and electromagnetic forces are mediated by massless, electrically neutral gluons and photons. The weak interaction bosons can carry an electric charge ($W^{\pm}$) and are rather massive, having around 90

[1]Do not confuse the chirality with *helicity*. The first one is constructed by $\frac{1\pm\gamma^5}{2}$ projection operators acting on Dirac spinors. The second one is just a projection of the spin on the direction of movement, $\hat{\Sigma}\hat{p}$, with $\hat{\Sigma}$ and $\hat{p}$ being the spin and momentum operators. Since the neutrinos are massive, the fixed helicity makes no sense: you can always find a rest frame, or a frame, where neutrino moves in the opposite direction, changing the sign of its momentum, but not the third component of spin.

Table 2: Interactions

Strong		Electromagnetic		Weak			
g (gluon)	m= 0 Q=0 s=1	γ (photon)	m= 0 Q=0 s=1	$W^{\pm}$	m= 80.4$[GeV]$ Q=±1 s=1	Z^0	m= 91.2$[GeV]$ Q=0 s=1

proton masses. Their massiveness is responsible for the short range of weak force. Originally the name "Standard Model" has been given to a theory of Weinberg, Salam ang Glashow and it has explained the structure of weak and electromagnetic interactions. The strong force is connected to a totally different symmetry group of the Lagrangian, then the other two and is an independent add-on. So far nobody has found a way to add the gravitational force to SM, so it is completely absent in these notes.

Last but not least is the Standard Model's Most Wanted: the scalar Higgs boson. It is responsible for the way SM is combining the weak and electromagnetic force as well as for the appearance of particle masses (with some problems around the neutrino mass generation, but it is beyond the scope of this part of the script). It has evaded all the experimental efforts of finding it so far, but at the end of 2011 there have been signals from the LHC experiment, that they may be close to finding it. If they succeed, then SM will become the most successful theory of elementary particles. It has already predicted the existence of neutral Z^0 bosons or the heavy top quark. Theoretical assumption confirmed by the later experiments!

2 How does it work?

Those, who have taken a look at the tables 1 and 2 may have alraedy noticed, that there are some simple rules concerning the electric charges of particles and their interactions. The "up" type quarks (u, c, t) have electric charges of $+2/3e$, whereas the "down" (d, s, b) type ones have charges of $-1/3e$. The charhge difference between each generation constituents is always e. The same rule applies to leptons. Electron, muon and taon have charges of $-e$ and neutrinos are always neutral. Weak $W^{\pm}$ bosons carry the $\pm$ charge and connect the "up" and "down" quarks as well as the charged leptons and neutrinos. For example in the β-decay in the nucleus a d quark changes into a u quark, emitting a virtual W^- boson, which decays into the e^- and $\overline{\nu}_e$. This kind of weak interaction is called "Charged Current" (CC). Neutral Current (NC) weak interaction was unknown before the introduction of Standard model. Anyway, the electric charge is always conserved. Furthermore the weak interaction connects only the left-handed particles, thus making the left-handed "up" and "down" or electron and neutrino pairs into some kind of doublets, leaving their right-handed parts as singlets (see fig. 2). And if we are talking about multiplets, conserved charges and currents we will come to a discussion about symmetries (Noether's theorem). Symmetries in particle physics come together with an appropriate **Lagrangian** containing the known phenomenological information. Here we need to generate four vector fields in order to reproduce the full electromagnetic and weak interaction pattern. This will require us to introduce local nonabelian gauge transformations for the left-handed doublets ($SU(2)$). And because the weak charged current interactions concern only the left-handed fields, one has

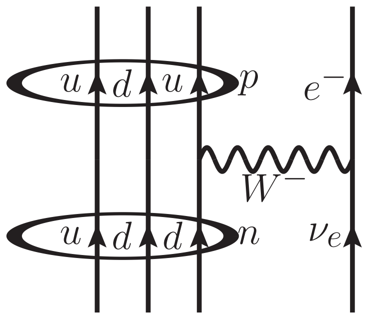

Figure 2: Neutron β-decay after the discovery of quarks and introduction of $W^{\pm}$ bosons.

to introduce an additional symmmetry group to incorporate the electromagnetic currents, which work with both chiralities. This will be the $U(1)$ "hypercharge" gauge. And in order to reproduce the right charges and weak boson masses one will have to break the initial symmetry. If you are confused with the "nonabelian gauge" and "symmetry breaking" in the field theory, the appendices A and B will be devoted for a little bit "handwaving" explanation of these problems. For more detailed discussion, containing step-by step derivations, see for example [3] ,[4].

2.1 The Standard Model of Electroweak Interactions

Let us remind, that the weak charge changing interaction connects only left-handed components of the local $SU(2)$ symmetry doublets. These contain "up" and "down" types of quarks or charged lepton and corresponding neutrino.

$$\psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \text{ or } \begin{pmatrix} l_L^- \\ \nu_{lL} \end{pmatrix} \tag{2.1}$$

The right-handed fields enter the Lagrangian as $SU(2)$ singlets. We would like to unify the weak and electromagnetic interactions. The problem of the $SU(2)$ group is that here it is responsible for the left-handed fermion interactions. The electromagnetic interaction connects both left- and right-handed fields. Thus we need to introduce an extra symmetry group $U(1)$ of the hypercharge Y in order to incorporate the electrodynamics as well!
We shall start from the free massless Dirac Lagrangian:

$$\mathcal{L}_0(x) = \sum_i i\overline{\psi}_L^i(x)\gamma^\mu\partial_\mu\psi_L^i(x) \tag{2.2}$$

with ψ_L^i representing the quark and lepton isodoublets 2.1. The index i stands here for the quark/lepton generation. Imposing the local gauge invariance on 2.2 will produce additional terms in the Lagrangian. The presence of local $SU(2)$ gauge will change derivatives acting on the left-handed fields (see section A or [3] or [4] for the details):

$$\partial_\mu\psi_L \to D_\mu^{SU(2)}\psi_L(x) = (\partial_\mu + ig\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu)\psi_L \tag{2.3}$$

Here $\boldsymbol{A}_\mu$ is the triplet of gauge fields A.17. The derivative acting on the right-handed components will not be affected. It is convenient to express the interaction Lagrangian connected to $SU(2)$ gauge symmetry as:

$$\mathcal{L}_{SU(2)}(x) = \left(-\frac{g}{2\sqrt{2}} j_\mu(x) W^\mu(x) + h.c.\right) - g j^3_\mu(x) A^{\mu 3}(x). \tag{2.4}$$

The first piece of current can be connected to the charged current interactions known from β-decays, pion physics or muon decays:

$$j_\mu(x) = \overline{\psi}_L(x)\gamma_\mu(\tau_1 + i\tau_2)\psi_L(x) = 2\overline{\psi}^+_L(x)\gamma_\mu\psi^-_L(x). \tag{2.5}$$

Here ψ^+_L and ψ^-_L are tha upper/lower components of the left-handed isotopic doublets.
The second part of our current conserves the charge:

$$j^3_\mu(x) = \overline{\psi}_L(x)\gamma_\mu\frac{1}{2}\tau_3\psi_L(x) = \frac{1}{2}\left(\overline{\psi}^+_L(x)\gamma_\mu\psi^+_L(x) - \overline{\psi}^-_L(x)\gamma_\mu\psi^-_L(x)\right). \tag{2.6}$$

In contrary to A.23 the above expression can not be identified with an electromagnetic current, because it lacks on the right-handed field components. Here the $U(1)_Y$ symmetry will come into play. It is an Abelian gauge symmetry like in Feynman's quantum electrodynamics, thus it will affect the right-handed field components as well:

$$\begin{aligned} \partial_\mu\psi^i_L &\rightarrow \left(\partial_\mu + ig\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu + ig'\frac{1}{2}Y^{doub}_L B_\mu\right)\psi^i_L \\ \partial_\mu\psi^{+i}_R &\rightarrow \left(\partial_\mu + ig'\frac{1}{2}Y^+_R B_\mu\right)\psi^{+i}_R \\ \partial_\mu\psi^{-i}_R &\rightarrow \left(\partial_\mu + ig'\frac{1}{2}Y^-_R B_\mu\right)\psi^{-i}_R \end{aligned} \tag{2.7}$$

We assume the $U(1)_Y$ gauge field B_μ to have different couplings to each of the lepton field components. The respective hypercharges are Y^{doub}_L for the left-handed isodoublets ψ^i_L, Y^+_R for the right-handed singlet of the "up"/ charged lepton component ψ^{+i}_R and Y^-_R for the right-handed singlet of the "down"/neutrino component ψ^{-i}_R.
In order to identify the hypercharges we shall make use of the Gell-Mann-Nishijima relation connecting the isotopic spin I_3 and hypercharge Y to the electric charge Q:

$$Q = I_3 + \frac{1}{2}Y. \tag{2.8}$$

and the phenomenological electric charges: $Q^{up} = \frac{2}{3}$ for the u, c, t quarks, $Q^{down} = -\frac{1}{3}$ for the d, s, b quarks, $Q^{e^-,\mu^-,\tau^-} = -1$ for the charged leptons and $Q^\nu = 0$ for all neutrinos. The

Table 3: Hypercharges

Field	q_L	u_R	d_R	l_L	l^-_R	ν_{lR}
Y	1/3	4/3	-2/3	-1	-2	0

hypercharges of isodoublets and singlets of fields have been summarized in table 3. With the hypercharge comes another conserved current:

$$j^Y_\mu = Y^{doub}_L\overline{\psi}_L\gamma_\mu\psi_L + Y^+_R\overline{\psi}^+_R\gamma\mu\psi^+_R + Y^-_R\overline{\psi}^-_R\gamma\mu\psi^-_R \tag{2.9}$$

which is connected to the total electromagnetic current by a relation:

$$j_\mu^{EM} = \frac{1}{2} j_\mu^Y + j_\mu^3. \tag{2.10}$$

From this definition one may construct a proper electromagnetic current connecting both left- and right-handed particle fields:

$$j_\mu^{EM} = Q^+ \overline{\psi}^+(x)\gamma_\mu \psi^+(x) + Q^- \overline{\psi}^-(x)\gamma_\mu \psi^-(x) \tag{2.11}$$

with $Q^\pm$ being the upper/lower isodoublet doublet component charges and $\psi^\pm(x) = \psi_L^\pm(x) + \psi_R^\pm(x)$. Thus the total interaction Lagrangian of the $SU(2) \times U(1)$ model may be written in the form:

$$\mathcal{L}_{SU(2)\times U(1)}(x) = \left(-\frac{g}{2\sqrt{2}} j_\mu(x) W^\mu(x) + h.c. \right) - j_\mu^3(x)(gA^{\mu 3}(x) - g' B^\mu) - g' j_\mu^{EM} B^\mu. \tag{2.12}$$

At this point we are actually ready to identify the interaction boson fields in the Lagrangian, but we still miss one important feature: the weak interaction boson masses. We shall break the gauge invariance introducing a Higgs field:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{2.13}$$

Here we have a $SU(2)$ doublet of complex charged ϕ^+ and neutral ϕ^0 fields with hypercharge $Y_\phi = 0$. The Higgs field is again described by a Lagrangian similar to B.2:

$$\begin{aligned} \mathcal{L} &= \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi) \\ V(\phi^\dagger \phi) &= -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2. \end{aligned} \tag{2.14}$$

By imposing the $SU(2) \times U(1)$ symmetry we shall couple the Higgs field to the gauge fields, just like in the case of quark and lepton isodoublets.

$$\partial_\mu \phi \to \left(\partial_\mu + ig\frac{1}{2}\boldsymbol{\tau} \cdot \boldsymbol{A}_\mu + ig' \frac{1}{2} B_\mu \right) \phi \tag{2.15}$$

The form of Higgs interaction potential 2.14 gives an infinite set of vacuums (states of field ϕ with the lowest possible energy), they satisfy the relation:

$$(\phi^+ \phi)_0 = \frac{v^2}{2} \tag{2.16}$$

with:

$$v^2 = \frac{\mu^2}{\lambda}. \tag{2.17}$$

The possible solutions for ϕ_0 differ only by a complex phase $e^{i\lambda}$. By choosing one of them the initial local $SU(2) \times U(1)$ symmetry gets spontaneously broken. This will generate mass terms of the gauge fields, as described shortly in section B or in more details in [3] and [4].

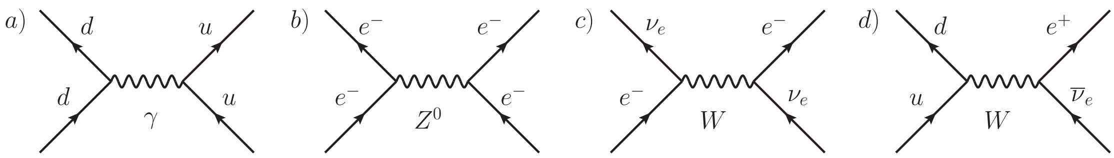

Figure 3: Example processes in the SM of electroweak interactions.

Using the equation 2.12 one can choose four lineary independent fields representing the physical $W^{\pm}$, Z^0 and γ bosons:

$$\begin{aligned} W^{\pm}_{\mu} &= \frac{1}{\sqrt{2}}\left(A^1_{\mu} \pm iA^2_{\mu}\right) \\ Z_{\mu} &= \frac{g}{\sqrt{g^2+g'^2}}A^3_{\mu} - \frac{g'}{\sqrt{g^2+g'^2}}B_{\mu} = \cos\Theta_W A^3_{\mu} - \sin\Theta_W B_{\mu} \\ A_{\mu} &= \frac{g'}{\sqrt{g^2+g'^2}}A^3_{\mu} + \frac{g}{\sqrt{g^2+g'^2}}B_{\mu} = \sin\Theta_W A^3_{\mu} + \cos\Theta_W B_{\mu}. \end{aligned} \tag{2.18}$$

Here we have introduced the weak angle Θ_W defined as:

$$\tan\Theta_w = \frac{g'}{g}. \tag{2.19}$$

The photon field A_μ remains massless, whereas the bosons W and Z as well as remainig scalar Higgs field H gain masses:

$$m^2_W = \frac{1}{4}g^2v^2,\ m^2_Z = \frac{1}{4}(g^2+g'^2)v^2,\ m^2_H = 2\mu^2. \tag{2.20}$$

The above results, published by S. Weinberg in [2] are fundamental for modern particle physics. Not only they unify the weak and electromagnetic interactions but they predict the existence of a particle unknown by that time: the Z boson and the weak NC interaction. The gauge bosons are connected to the interactions between fermions and quarks. An example can be seen in the figure 3: a) electromagnetic scattering, b) neutral current scattering, c), d) charged current scattering. The interaction Lagrangian of SM can be expressed in the terms of physical fields (2.18).

$$\mathcal{L}_{I\ SM} = \left(-\frac{g}{2\sqrt{2}}j^{CC}_{\mu}W^{\mu} + h.c.\right) - \frac{g}{2\cos\Theta_W}j^{NC}_{\mu}Z^{\mu} - ej^{EM}_{\mu}A^{\mu}. \tag{2.21}$$

The corresponding charged currents are:

$$\begin{aligned} j^{CC}_{\mu} &= 2\overline{u}_L\gamma_\mu d_L \ (\text{"}up\text{" and "}down\text{" quarks}) \\ j^{CC}_{\mu} &= 2\overline{\nu}_{lL}\gamma_\mu l_L \ (neutrinos\ and\ leptons) \end{aligned} \tag{2.22}$$

the electromagnetic currents:

$$\begin{aligned} j^{EM}_{\mu} &= \sum_{q=u,d,c,s,t,b} e_q\overline{q}\gamma_\mu q \ (\text{"}up\text{" and "}down\text{" quarks}) \\ j^{EM}_{\mu} &= -\bar{l}\gamma_\mu l \ (charged\ leptons) \end{aligned} \tag{2.23}$$

with e_q being the quark charge: 2/3 for u, c, t and $-1/3$ for d, s, b. Notice, that the neutrinos ,as neutral particles, do not have any electromagnetic interactions. Finally, the neutral current is defined as:

$$j_\mu^{NC} = 2j_\mu^3 - 2\sin^2\Theta_W j_\mu^{EM} \tag{2.24}$$

both for quarks and leptons. With these prescriptions one is capable of calculating all processes concerning the weak and electromagnetic interactions (examples in fig. 3) of particles listed in table 1.

The Higgs boson is also responsible for generation of particle masses. The weak gauge boson masses can be deduced from the interaction Lagrangian 2.21 and from the low-energy limit of the charged current interaction 1.2, which occures for four-momentum transfers $Q^2 \ll m_W^2$.

$$\begin{aligned}\frac{m_W}{m_Z} &\approx \cos\Theta_W \qquad (2.25)\\ m_W &= \left(\frac{\pi\alpha}{\sqrt{2}G_F}\right)^{\frac{1}{2}} \frac{1}{\sin\Theta_W(1-\Delta r)}\\ m_Z &= \left(\frac{\pi\alpha}{\sqrt{2}G_F}\right)^{\frac{1}{2}} \frac{1}{\sin\Theta_W\cos\Theta_W}\end{aligned}$$

I the above equations $G_F = 1.6637(1)\cdot 10^{-5}GeV^{-2}$ is the Fermi coupling constant, $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$- the fine structure constant and $\Delta r \approx 0.07$ are the so-called "radiative corrections" for the W boson. The Wienberg weak angle is also known with good precision. Using these relations one can reproduce the values of boson masses listed in table 1.

The quark and lepton mass generation mechanism through coupling to the Higgs field after the symmetry breaking is another story: these masses are not constrained, as in the case of the W and Z masses and are free parameters in the model. There is one significant result coming from the higgs quark and lepton mass generation mechanism: quark and lepton fields entering in the charged currents (2.22) are no longer the same, as the fields appearing in the free Lagrangian A.1. They appear as an unitary combination of the free fields, for example quarks:

$$d_L^{i\ mix}(x) = \sum_{d'=d,s,b} V_{ud'} d'_L(x) \tag{2.26}$$

with $d^i \in (d,s,b)$ and the unitary ($V^\dagger V = 1$) mixing matrix being the famous Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix. The same phenomenon happens to the leptons. A very interesting question is, whether the neutrinos can acquire mass through the Higgs mechanism. It is very unlikely, because their masses are unexpectedly small compared to the other spin-1/2 particles. But the neutrino state mixing has been proven by the so-called "neutrino oscillations"! The only question is whether this mixing has the same origin, as in the quark case?

3 Summary

The Standard Model of the electroweak interaction is based on few principles:

1. The Lagrangian of massless fields is invariant under the local $SU(2)_L \times U(1)_Y$ gauge.

2. The weak and electromagnetic interactions are unified.

3. The masses of the particles are generated through the Higgs mechanism.

This theory has been very successful in describing processes as the β-decay, μ-decay, π-decay, decays of the strange particles, neutrino interactions and many more. It has also one of the biggest predictive powers in physics. It predicted the existence of Z^0 bosons and neutral currents. It is also renormalizable (all infinities appearing in the higher orders of perturbation expansion can be emoved). But there is one condition of its renormalizability: sum of all the charges of fields entering $SU(2)$ doublets must be equal to zero. Thus:

$$3\left(\frac{2}{3}-\frac{1}{3}\right)N_f^q - N_f^l = 0 \tag{3.1}$$

with N_f^q and N_f^l being the numbers of quarks and lepton families. From the above relation one can see, that:

$$N_f^q = N_f^l. \tag{3.2}$$

Historically the third family of leptons was unknown at the time the SM has been formulated ([2]). But in 1975 the τ lepton has been discovered. This implied the existence of the ν_τ as well as the existence of the third generation of quarks t and b. All of it has been later proven experimentally!

A Local SU(2) Gauge Invariance

Let us start with a free Lagrangian of the fermionic (no information about the helicity yet!) doublets:

$$\mathcal{L}_0(x) = \overline{\psi}(x)\left(i\gamma^\mu\partial_\mu + m\right)\psi(x) \tag{A.1}$$

with

$$\psi(x) = \begin{pmatrix} \psi^+(x) \\ \psi^-(x) \end{pmatrix} \tag{A.2}$$

being a fermionic field doublet. But free field theories are simply boring. We would like to put some interactions and currents in our model. In order to do so we will introduce a continuous symmetry transformation of the field doublet $\psi(x)$. One of the minimal nontrivial (mixing the + and − components) group of such transformations is the $SU(2)$ group. Let us strart with a global transformation:

$$\psi'(x) = U\psi(x),\ \overline{\psi}'(x) = \overline{\psi}U^\dagger,\ U = exp(\frac{i}{2}\boldsymbol{\tau\zeta}) \tag{A.3}$$

with τ being a vector of Pauli matrices (generators of the $SU(2)$ rotations of two-component fields) and $\boldsymbol{\zeta}$ being three real real rotation parameters. In some sense this transformation works like a rotation of a two-component complex vector using a 2×2 matrix. It is simple to show, that our $\mathcal{L}_0$ is invariant under such transformation and the conserved "isovector" current reads:

$$j_i^\mu(x) = \overline{\psi}(x)\gamma^\mu\frac{1}{2}\tau_i\psi(x),\ \partial_\mu j_i^\mu(x) = 0 \tag{A.4}$$

The total isotopic spin $T_i = \int d^3x j_i^0(x)$ is another conserved number. The global symmetry is lacking on one very important feature: it does not generate any interactions of our fields. Now we shall assume, that the set of rotation parameters $\boldsymbol{\zeta}$ is depending on x:

$$U \to U(x) = exp(\frac{i}{2}\boldsymbol{\tau}\boldsymbol{\zeta}(x)) \tag{A.5}$$

The scalar part $m\overline{\psi}(x)\psi(x)$ of the previous Lagrangian is again trivially invariant under such transformation, but the derivative part isn't. It can be shown, that:

$$\partial_\mu \psi(x) = U^\dagger(x)\left(\partial_\mu - \frac{i}{2}\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\zeta}(x)\right)\psi'(x) \tag{A.6}$$

We want to have invariance under the local $SU(2)$ transformations. There is one way to restore it: one has to add an extra vector field satisfying the relation:

$$A_i^\mu(x) = \frac{1}{g}\partial^\mu\zeta_i(x) \tag{A.7}$$

with g being a dimensionless coupling constant. This field wll interact with our fermionic fields through the so- called "covariant derivative":

$$D_\mu\psi(x) = \left(\partial_\mu + ig\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu(x)\right)\psi(x) \tag{A.8}$$

By considering the general $SU(2)$ transformation properties one can show, that:

$$D'_\mu = U(x)D_\mu U^\dagger(x) = \partial_\mu + iq\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{A}'_\mu(x) \tag{A.9}$$
$$\boldsymbol{A}'_\mu(x) = \boldsymbol{A}_\mu(x) - \frac{1}{g}\partial_\mu\boldsymbol{\zeta}(x) - \boldsymbol{\zeta}(x)\times\boldsymbol{A}_\mu(x)$$

which leads to the following transformation of the whole derivative term in Lagrangian:

$$D_\mu\psi(x) = U^\dagger(x)D'_\mu\psi'(x) \tag{A.10}$$

making the following Lagrangian invariant under local $SU(2)$:

$$\mathcal{L}_{SU(2)}(x) = \overline{\psi}(x)\left(i\gamma^\mu D_\mu + m\right)\psi(x) \tag{A.11}$$

This is a step in the righ direction: our fermionic field doublet is interacting with external vector field $\boldsymbol{A}_\mu(x)$. Now we would like to know, how to describe the dynamics of the above mentioned vector field. Let us consider the commutator of covariant derivatives:

$$[D_\mu, D_\beta] = ig\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{F}_{\mu\beta}(x) \tag{A.12}$$

where the nonabelian vector field strength is given by:

$$\boldsymbol{F}_{\mu\beta}(x) = \partial_\mu\boldsymbol{A}_\beta(x) - \partial_\beta\boldsymbol{A}_\mu(x) - g\boldsymbol{A}_\mu(x)\times\boldsymbol{A}_\beta(x) \tag{A.13}$$

Its transformation properties under the local $SU(2)$ is as follows:

$$\boldsymbol{F}'_{\mu\beta}(x) = \boldsymbol{F}_{\mu\beta}(x) - \boldsymbol{\zeta}(x)\times\boldsymbol{F}_{\mu\beta}(x) \tag{A.14}$$

which can be shown just by doing the known transformations (eq. A.10). Having this identity we see, that the contraction

$$\boldsymbol{F}_{\mu\beta}(x) \cdot \boldsymbol{F}^{\mu\beta}(x) \tag{A.15}$$

is also $SU(2)$ invariant, giving us the proper field strength. The whole Lagrangian with minimal coupling[2] is as follows:

$$\mathcal{L}(x) = \overline{\psi}(x)\,(i\gamma^\mu D_\mu + m)\,\psi(x) - \boldsymbol{F}_{\mu\beta}(x) \cdot \boldsymbol{F}^{\mu\beta}(x) \tag{A.16}$$

It includes now the gauge fields coupled to fermionic isotopic current:

$$\mathcal{L}_I(x) = -g\boldsymbol{j}_\mu(x) \cdot \boldsymbol{A}^\mu(x) = -g\sum_{i=1}^{3} j_{i\mu(x)} A_i^\mu(x) \tag{A.17}$$

where:

$$j_{i\mu}(x) = \overline{\psi}(x)\gamma_\mu \frac{1}{2}\tau_i \psi(x). \tag{A.18}$$

For the sake of physical clearness it is convenient to introduce the following notation:

$$j_\mu(x) = 2(j_\mu^1 + ij_\mu^2);\; W_\mu(x) = \frac{1}{\sqrt{2}}(A_\mu^1 - iA_\mu^2). \tag{A.19}$$

The interaction Lagrangian may be re-written as:

$$\mathcal{L}_I(x) = \left(-\frac{g}{2\sqrt{2}} j_\mu(x) W^\mu(x) + h.c.\right) - g j_\mu^3(x) A^{\mu 3}(x). \tag{A.20}$$

Here we need to do some remarks about our currents:

$$j_\mu(x) = \overline{\psi}(x)\gamma_\mu(\tau_1 + i\tau_2)\psi(x) = 2\overline{\psi}^+(x)\gamma_\mu\psi^-(x). \tag{A.21}$$

The isotopic doublet components ψ^+ and ψ^- have the third projections of the isotopic spin I_3 equal to $1/2$ and $-1/2$ respectively. From the Gell-Mann-Nishijima relation the electric charge Q, isotopic spin projection I_3 and hypercharge Y are related through:

$$Q = I_3 + \frac{1}{2}Y. \tag{A.22}$$

Thus the current $j_\mu(x)$ changes the particle charges by one. And the vector field $W^\mu(x)$ can be identified through electric charge conservation with the charged $W^\pm$ boson field. The remaining piece of current should be electrically neutral. Indeed:

$$j_\mu^3(x) = \overline{\psi}(x)\gamma_\mu \frac{1}{2}\tau_3\psi(x) = \frac{1}{2}\left(\overline{\psi}^+(x)\gamma_\mu\psi^+(x) - \overline{\psi}^-(x)\gamma_\mu\psi^-(x)\right) \tag{A.23}$$

and the field $A^3(x)$ is the field of neutral vector particles.
By introducing the local $SU(2)$ gauge invariance we have built a theory of fermions interacting both with charged and neutral vector fields. It is a good starting point for the theory of weak and electromagnetic ineractions. As usual, there are few "buts":

[2] Of course, there are more complicated possibilities, like the tensor interaction $\mathcal{L}_T = \mu\overline{\psi}(x)\sigma_{\mu\beta}\boldsymbol{\tau}\psi(x)\boldsymbol{F}^{\mu\beta}(x)$, but let us not get the things too complicated here. Just bear in mind, that the local gauge invariance does not limit the coupling to minimal case.

1. Compare the tables 1, 2 with the equation A.16. Aren't we missing something important? Indeed, the weak vector bosons are rather massive! But the mass term $-m_v^2 \boldsymbol{A}^\mu \boldsymbol{A}_\mu$ will break the $SU(2)$ symmetry, which we have introduced with such a big effort.
2. In the introduction we have mentioned the fact, that there are only left-handed neutrinos. Our Lagrangian needs to be modified in order to account for that.

The next section will be devoted to solving first of the problems: how to break the gauge symmetry in a convenient way so that we end up with physically meaningful results?

B Spontaneous Symmetry Breaking and the Higgs Mechanism

Let us consider the scalar complex field $\phi(x)$ with the Lagrangian:

$$\begin{aligned} \mathcal{L} &= \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi) \\ V(\phi^\dagger \phi) &= -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \end{aligned} \tag{B.1}$$

notice, that the quadratic potential is not a mass term, the sign is different! The Lagragian is invariant under a global transformation

$$\phi'(x) = e^{i\zeta} \phi(x) \tag{B.2}$$

. The system described by the above Lagrangian (B.2) reaches the minimal energy for

$$\phi_0^\dagger \phi_0 = \frac{\mu^2}{2\lambda} \tag{B.3}$$

(just write down the potential as $V(\phi^\dagger\phi) = \lambda \left(\phi^\dagger \phi - \frac{\mu^2}{2\lambda}\right)^2 - \frac{\mu^4}{4\lambda}$). This solution has two distinct features:

- The minimal energy is different from zero.
- The number of solutions for minimal energy is greater, than one. Actually it is infinite:

$$\phi_0 = \frac{v}{\sqrt{2}} e^{i\eta}, \ v = \frac{\mu}{\sqrt{\lambda}} \tag{B.4}$$

In the above expression $e^{i\eta}$ is an arbitrary complex phase. We need to choose our vacuum. This choice is arbitrary, let it simply be:

$$\phi_0 = \frac{v}{\sqrt{2}}. \tag{B.5}$$

But now we break the symmetry given by B.2. If the initial symmetry is broken by a choice of vacuum we are talking about the "spontaneous symmetry breaking". Let us now introduce two real fields by the decomposition of our complex scalar field:

$$\phi(x) = \frac{v}{\sqrt{2}} + \frac{\chi_1 + i\chi_2}{\sqrt{2}} \tag{B.6}$$

The new fields χ_1 and χ_2 have zero vacuum expectation values. Now one can re-write the Lagrangian in the following form:

$$\mathcal{L} = \frac{1}{2}\sum_i \partial_\mu \chi_i \partial^\mu \chi_i - \frac{1}{2}2\mu^2\chi_1^2 - \lambda v \chi_1(\chi_1^2 + \chi_2^2) - \frac{\lambda}{4}(\chi_1^2 + \chi_2^2)^2. \tag{B.7}$$

The most important result up to this point is that the field χ_1 has now a mass term $-\frac{1}{2}2\mu^2\chi_1^2$, whereas the second field remains massless. This is the key point of the so-called "Goldstones theorem": if generic continuous symmetry which spontaneously broken, then new massless (or light, if the symmetry is not exact) scalar particles appear in the spectrum of possible excitations. This is actually a problem of such theories, because no massless scalar particles have ever been observed experimentally. On the other hand in local gauge invariant theories all vector gauge bosons are massless. But Higgs has shown, that if we take both type of theories and mix them, we can get disadvantage × disadvantage = advantage! How to perform such a trick?
Let us start with a theory of a complex scalar field invariant under a local gauge transformation. For simplicity let it ba an Abelian gauge ($U(1)$ gauge, like the one generating the electromagnetic field in Feynman's quantum electrodynamics):

$$\phi'(x) = e^{i\zeta(x)}\phi(x) \tag{B.8}$$

This one will generate a real vector field, which will lead (together with a self interaction of scalar field of the form B.2 and the vector field strength tensor $F_{\mu\beta} = \partial_\mu A(x)_\beta - \partial_\beta A(x)_\mu$):

$$\mathcal{L}(x) = ((\partial_\mu + igA_\mu(x))\phi(x))^\dagger\left((\partial^\mu + igA^\mu(x))\phi(x)\right) - V(\phi^\dagger\phi) - \frac{1}{4}F_{\mu\beta}F^{\mu\beta}. \tag{B.9}$$

Again, we have a spontaneous symmetry breaking pattern, in which the generic solution minimizing the scalar field energy is given by:

$$\phi_0 = \frac{v}{\sqrt{2}}e^{i\eta}. \tag{B.10}$$

We shall break the symmetry by choosing $\eta = 0$. The complex field $\phi(x)$ be represented in the following form:

$$\phi(x) = \frac{v + \chi(x)}{\sqrt{2}}e^{i\omega(x)} \tag{B.11}$$

where $\chi(x)$ and $\omega(x)$ are real functions. Because of some freedom in the choice of gauge $\zeta(x)$ (B.8), the local complex phase $\omega(x)$ may be absorbed. This is the so-called "unitary gauge".
We may combine now (B.9) and (B.11) in order to re-express the Lagrangian:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}2\mu^2\chi^2 + \frac{1}{2}g^2v^2A_\mu A^\mu + \frac{1}{2}g^2(2v\chi + \chi^2)A_\mu A^\mu + \\ &- \frac{1}{2}\lambda(4v\chi^2 + \chi^4) - \frac{1}{4}F_{\mu\beta}F^{\mu\beta}\end{aligned} \tag{B.12}$$

The above expression is very fundamental for particle physics. Term $\frac{1}{2}g^2v^2A_\mu A^\mu$ has geerated a mass term of the vector field and $\frac{1}{2}2\mu^2\chi^2$ gives the mass to the scalar field. We have

arrived at the famous "Higgs mechanism" which generates masses of the vector gauge fields. There are no massless Goldstone fields anymore. You can say, that in the Higgs mechanism after spontaneous symmetry breaking gauge fields "eat" the Goldstone bosons, becoming massive. It can be shown on the number of degrees of freedom. Before the symmetry breaking we've had a complex field with two components and a massless vector field with two independent degrees of freedom, polarizations (2+2). After the symmetry breaking the Goldstone mode of scalar field has become an additional degree of freedom ofthe vector field. This is due to the fact, that massive vector fields have an additional, longitudinal, polarization. And we have ended with 1 scalar + 3 vector degrees of freedom.
At this point wee know, how to make a theory with neutral and charged gauge fields and how to give them a mass (at least in the Abelian gauge transformation case, but the generalization is only a small technical complication now). We should be ready now to construct the full model of the weak+electromagnetic interactions.

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